Decomposition of the deformations of a thin shell. Nonlinear elastic models

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Abstract. We investigate the behavior of the deformations of a thin shell, whose thickness δ tends to zero, through a decomposition technique of these deformations. The terms of the decomposition of a deformation v are estimated in terms of the L^2-norm of the distance from ∇v to $SO(3)$. This permits in particular to derive accurate nonlinear Korn's inequalities for shells (or plates). Then we use this decomposition technique and estimates to justify a nonlinear bending model for elastic shells for an elastic energy of order δ^3.

1. Introduction

The concern of this paper is twofold. We first give a decomposition technique for the deformation of a shell which allows to established a nonlinear Korn type inequality for shells. In a second part of the paper, we use such a decomposition to derive a nonlinear elastic shell model.

In the first part, we introduce two decompositions of an admissible deformation of a shell (i.e. which is H^1 with respect to the variables and is fixed on a part of the lateral boundary) which take into account the fact that the thickness 2δ of such a domain is small. This decomposition technique has been developed in the framework of linearized elasticity for thin structures in [17], [18], [19] and for thin curved rods in nonlinear elasticity in [5]. As far as large deformations are concerned these decompositions are obtained through using the "Rigidity Theorem" proved in [14] by Friesecke, James and Müller together with the geometrical precision of this result given in [5]. Let us consider a shell with mid-surface S and thickness 2δ. The two decompositions of a deformation v defined on this shell are of the type

$$v = \mathcal{V} + s_3\mathbf{R}\mathbf{n} + \overline{v}.$$

In the above expression, the fields $\mathcal{V}$ and $\mathbf{R}$ are defined on S, s_3 is the variable in the direction $\mathbf{n}$ which is a unit vector field normal to S and $\overline{v}$ is a field still defined on the 3D shell. Let us emphasize that the terms of the decompositions $\mathcal{V}$, $\mathbf{R}$ and $\overline{v}$ have at least the same regularity than v and satisfy the corresponding boundary conditions. Loosely speaking, the two first terms of the decompositions reflect the mean of the deformation over the thickness and the rotations of the fibers of the shell in the direction $\mathbf{n}$. For the above decomposition, it worth noting that the fields $\mathcal{V}$, $\mathbf{R}$ and $\overline{v}$ are estimated in terms of the "energy" $||dist(\nabla_x v, SO(3))||_{L^2(S\times]-\delta,\delta[)}$ and the thickness of the shell.

In the first decomposition, the field $\mathbf{R}$ satisfies

$$||dist(\mathbf{R}, SO(3))||_{L^2(S)} \le \frac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(S\times]-\delta,\delta[)}$$

which shows that the field $\mathbf{R}$ is close to a rotation field for small energies.

In the second decomposition, for which we assume from the beginning that $||dist(\nabla_x v, SO(3))||_{L^2} \leq C(S)\delta^{3/2}$ where $C(S)$ is a geometrical constant, the field $\mathbf{R}$ is valued in $SO(3)$.

For thin structures, the usual technique in order to rescale the applied forces to obtain a certain level of energy is to established nonlinear Korn's type inequalities. Using Poincaré inequality as done in [16] (see also [10] and Subsection 4.1 of the present paper) leads in the case of a shell to the following inequality

$$||v - I_d||_{(L^2(S\times]-\delta,\delta[))^3} + ||\nabla_x v - \mathbf{I}_3||_{(L^2(S\times]-\delta,\delta[))^9} \leq C(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(S\times]-\delta,\delta[)}).$$

The first important consequence of the decomposition technique together with its estimates is the following nonlinear Korn's inequality for shells

$$||v - I_d||_{(L^2(S\times]-\delta,\delta[))^3} + ||\nabla_x v - \mathbf{I}_3||_{(L^2(S\times]-\delta,\delta[))^9} \leq \frac{C}{\delta}||dist(\nabla_x v, SO(3))||_{L^2(S\times]-\delta,\delta[)}$$

Indeed the two inequalities identify for energies of order $\delta^{3/2}$ which is the first interesting critical case. For smaller levels of energy, the second estimate is more relevant. We also establish the following estimate for the linear part of the strain tensor

$$\big\|\nabla_x v+(\nabla_x v)^T-2\mathbf{I}_3\big\|_{(L^2(S\times]-\delta,\delta[))^9} \leq C||dist(\nabla_x v, SO(3))||_{L^2(S\times]-\delta,\delta[)}\Big\{1+\frac{1}{\delta^{5/2}}||dist(\nabla_x v, SO(3))||_{L^2(S\times]-\delta,\delta[)}\Big\}$$

which shows that $||dist(\nabla_x v_\delta, SO(3))||_{L^2(S\times]-\delta,\delta[)} \sim \delta^{5/2}$ is another critical case. For such level of energy, our Korn's inequality for shells turns out to appear as an important tool. We have established and used the analogue of these inequalities for rods in [5].

In the present paper we focus on the case where $||dist(\nabla_x v_\delta, SO(3))||_{L^2(S\times]-\delta,\delta[)} \sim \delta^{3/2}$ which is the highest level of energy which can be analyzed through our technique. The lower level of energies will be studied in a forthcoming paper.

For $||dist(\nabla_x v_\delta, SO(3))||_{L^2(S\times]-\delta,\delta[)} \sim \delta^{3/2}$, we deduce the expression of the limit of the Green-St Venant's strain tensor from the decompositions, the associated estimates and a standard rescaling and the result is the same using the two decompositions.

In the second part of the paper, we strongly use the results of the first part in order to derive limit 2D shells models. As a general reference on elasticity theory we refer to [6] and we start from a total energy $\displaystyle\int \widehat{W}(\nabla v) - \int f\cdot v$ where $\widehat{W}$ is the local elastic energy. We assume that

$$\widehat{W}(\nabla v) = \begin{cases} W\big((\nabla v)^T\nabla v - \mathbf{I}_3\big) & \text{if } \det(\nabla v) > 0, \\ +\infty & \text{if } \det(\nabla v) \leq 0. \end{cases}$$

The assumptions on the function W are similar to those adopted in [14]. We assume that W is continuous from $\mathbf{S}_3$ (the space of symmetric 3×3 matrices) into $\mathbb{R}$ and that

$$\begin{aligned} &\exists c > 0 \quad \text{such that} \quad \forall E \in \mathbf{S}_3 \quad W(E) \geq c|||E|||^2, \\ &\forall \varepsilon > 0, \quad \exists \theta > 0, \quad \text{such that} \quad \forall E \in \mathbf{S}_3 \quad |||E||| \leq \theta \implies |W(E) - Q(E)| \leq \varepsilon|||E|||^2, \end{aligned}$$

where Q is a positive quadratic form. Recall that $\widehat{W}(\nabla v) \geq c(dist(\nabla v, SO(3))^2$.

Using the nonlinear Korn's inequality for shells mentioned above, we are in a position to scale the applied forces f in order to obtain a given order of total energy (with respect to δ) . Then we deduce the order of the quantity $||dist(\nabla_x v_\delta, SO(3))||_{L^2(S\times]-\delta,\delta[)}$ with respect to the scaling of the forces in the general case. In the following, we choose the applied forces so that $||dist(\nabla_x v_\delta, SO(3))||_{L^2(S\times]-\delta,\delta[)} \sim \delta^{3/2}$.

At last we derive the "limit" energy as δ goes to 0 using a Γ-limit procedure on sequences of fields $\mathcal{V}_\delta$, $\mathbf{R}_\delta$ and $\overline{v}_\delta$. Through a possible elimination of $\overline{v}$ in the Γ-limit energy, we finally obtain a minimization problem for the mean deformation $\mathcal{V}$ and the rotation $\mathbf{R}$ under a constraint between $\nabla\mathcal{V}$ et $\mathbf{R}$.

As general references on the theory of nonlinear elasticity, we refer to [1], [6] and [22] and to the extensive bibliographies of these works. For the justification of plates or shell models in nonlinear elasticity we refer to [7], [8], [9], [11], [12], [15], [19], [21], [23], [24]. A general introduction of Γ-convergence can be found in [13]. The rigidity theorem and its applications to thin structures using Γ-convergence arguments are developed in [14], [15], [20], [21]. The decomposition of the deformations in thin structures is introduced in [17], [18] and a few applications to the junctions of multi-structures and homogenization are given in [2], [3], [4].

The paper is organized as follows. Section 2 is devoted to describe the geometry of the shell and to give a few notations. In Section 3 we introduce the two decompositions of the deformations of a thin shell and we derive the estimates on the terms of these decompositions. We precise the boundary conditions on the deformation and we establish a nonlinear Korn's inequality for shells in Section 4. Section 5 is concerned with a standard rescaling. We derive the limit of the Green-St Venant strain tensor of a sequence of deformations such that $||dist(\nabla_x v_\delta, SO(3))||_{L^2(S\times]-\delta,\delta[)} \sim \delta^{3/2}$ in Section 6. In Section 7 we consider nonlinear elastic shells and we use the results of the proceeding sections to scale the applied forces in order to obtain a priori estimates on the deformation. In Section 8 is devoted to derive the Γ-limit for energies of order $\delta^{3/2}$. At last two appendices contain a few technical results on the interpolation of rotations and an algebraic elimination for quadratic forms.

2. The geometry and notations.

Let us introduce a few notations and definitions concerning the geometry of the shell (see [17] for a detailed presentation).

Let ω be a bounded domain in $\mathbb{R}^2$ with lipschitzian boundary and let ϕ be an injective mapping from $\overline{\omega}$ into $\mathbb{R}^3$ of class $\mathcal{C}^2$. We denote S the surface $\phi(\overline{\omega})$. We assume that the two vectors $\dfrac{\partial\phi}{\partial s_1}(s_1,s_2)$ and $\dfrac{\partial\phi}{\partial s_2}(s_1,s_2)$ are linearly independent at each point $(s_1,s_2)\in\overline{\omega}$.

We set

$$\mathbf{t}_1 = \frac{\partial\phi}{\partial s_1},\qquad \mathbf{t}_2 = \frac{\partial\phi}{\partial s_2},\qquad \mathbf{n} = \frac{\mathbf{t}_1\wedge\mathbf{t}_2}{\big\|\mathbf{t}_1\wedge\mathbf{t}_2\big\|_2}. \tag{2.1}$$

The vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are tangential vectors to the surface S and the vector $\mathbf{n}$ is a unit normal vector to this surface. The reference fiber of the shell is the segment $]-\delta,\delta[$. We set

$$\Omega_\delta = \omega\times]-\delta,\delta[.$$

Now we consider the mapping $\Phi : \overline{\omega}\times\mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$\Phi \;:\; (s_1,s_2,s_3) \longmapsto x = \phi(s_1,s_2) + s_3\mathbf{n}(s_1,s_2). \tag{2.2}$$

There exists $\delta_0\in(0,1]$ depending only on S, such that the restriction of Φ to the compact set $\overline{\Omega}_{\delta_0} = \overline{\omega}\times[-\delta_0,\delta_0]$ is a $\mathcal{C}^1-$ diffeomorphism of that set onto its range (see e.g. [8]). Hence, there exist two constants $c_0>0$ and $c_1\ge c_0$, which depend only on ϕ, such that

$$\forall\delta\in(0,\delta_0],\qquad \forall s\in\Omega_{\delta_0},\qquad c_0\le |||\nabla_s\Phi(s))|||\le c_1,\quad \text{and for } x=\Phi(s)\qquad c_0\le |||\nabla_x\Phi^{-1}(x))|||\le c_1.$$

Definition 2.1. *For $\delta \in (0, \delta_0]$, the shell $\mathcal{Q}_\delta$ is defined as follows:*

$$\mathcal{Q}_\delta = \Phi(\Omega_\delta).$$

The mid-surface of the shell is S. The lateral boundary of the shell is $\Gamma_\delta = \Phi(\partial\omega\times]-\delta,\delta[)$. The fibers of the shell are the segments $\Phi\big(\{(s_1,s_2)\}\times]-\delta,\delta[\big)$, $(s_1,s_2)\in\omega$. We respectively denote by x and s the running points of $\mathcal{Q}_\delta$ and of Ω_δ. A function v defined on $\mathcal{Q}_\delta$ can be also considered as a function defined on Ω_δ that we will also denote by v. As far as the gradients of v are concerned we have $\nabla_x v$ and $\nabla_s v = \nabla_x v.\nabla\Phi$ and e.g. for a.e. $x = \Phi(s)$

$$c|||\nabla_x v(x)||| \le |||\nabla_s v(s)||| \le C|||\nabla_x v(x)|||,$$

where the constants are strictly positive and do not depend on δ.

Since we will need to extend a deformation defined over the shell $\mathcal{Q}_\delta$, we also assume the following.

For any $\eta > 0$, let us first denote the open set

$$\omega_\eta = \big\{(s_1,s_2)\in\mathbb{R}^2 \mid dist\big((s_1,s_2),\omega\big) < \eta\big\}.$$

We assume that there exist $\eta_0 > 0$ and an extension of the mapping ϕ (still denoted ϕ) belonging to $\big(\mathcal{C}^2(\overline{\omega}_{\eta_0})\big)^3$ which remains injective and such that the vectors $\dfrac{\partial\phi}{\partial s_1}(s_1,s_2)$ and $\dfrac{\partial\phi}{\partial s_2}(s_1,s_2)$ are linearly independent at each point $(s_1,s_2)\in\overline{\omega}_{\eta_0}$. The function Φ (introduced above) is now defined on $\overline{\omega}_{\eta_0}\times[-\delta_0,\delta_0]$ and we still assume that it is a $\mathcal{C}^1-$ diffeomorphism of that set onto its range. Then there exist four constants $c_0^{'}$, $c_1^{'}$, $c^{'}$ and $C^{'}$ such that

$$\begin{cases}\forall s\in\overline{\omega}_{\eta_0}\times[-\delta_0,\delta_0], \qquad c_0^{'}\le |||\nabla_s\Phi(s)|||\le c_1^{'}, \ \text{ and for } \ x=\Phi(s) \quad c_0^{'}\le|||\nabla_x\Phi^{-1}(x)|||\le c_1^{'}\\ c^{'}|||\nabla_x v(x)|||\le|||\nabla_s v(s)|||\le C^{'}|||\nabla_x v(x)|||, \quad \text{for a.e. } x=\Phi(s).\end{cases} \tag{2.3}$$

At the end we denote by I_d the identity map of $\mathbb{R}^3$.

3. Decompositions of a deformation.

In this Section, we recall the theorem of rigidity established in [14] (Theorem 3.1 of Section 3.1). In Subsection 3.2 we recall that any deformation can be extended in a neighborhood of the lateral boundary of the shell with the same level of "energy". Then we apply Theorem 3.1 to a covering of the shell. In Subsections 3.4 and 3.5, we introduce the two decompositions of a deformation and we established estimates on these decompositions in term of $||dist(\nabla_x v, SO(3))||_{L^2}$.

3.1. Theorem of rigidity.

We equip the vector space $\mathbb{R}^{n\times n}$ of $n\times n$ matrices with the Frobenius norm defined by

$$\mathbf{A} = \big(a_{ij}\big)_{1\le i,j\le n}, \qquad\qquad |||\mathbf{A}||| = \sqrt{\sum_{i=1}^n\sum_{j=1}^n |a_{ij}|^2}.$$

We just recall the following theorem established in [14] in the version given in [5].

Theorem 3.1. *Let Ω be an open set of $\mathbb{R}^n$ contained in the ball $B\big(O;R\big)$ and star-shaped with respect to the ball $B\big(O;R_1\big)$ $(0 < R_1 \le R)$. For any $v \in \big(H^1(\Omega)\big)^n$, there exist $\mathbf{R}\in SO(n)$ and $\mathbf{a}\in\mathbb{R}^n$ such that*

$$\begin{cases}||\nabla_x v - \mathbf{R}||_{(L^2(\Omega))^{n\times n}} \le C||dist(\nabla_x v; SO(n))||_{L^2(\Omega)},\\ ||v-\mathbf{a}-\mathbf{R}\,x||_{(L^2(\Omega))^n}\le CR||dist\big(\nabla_x v; SO(n)\big)||_{L^2(\Omega)},\end{cases} \tag{3.1}$$

where the constant C depends only on n and $\dfrac{R}{R_1}$.

3.2. Extension of a deformation and splitting of the shell.

In order to make easier the decomposition of a deformation as the sum of an elementary deformation given via an approximate field of rotations (see Subsection 3.4) or a field of rotations (see Subsection 3.5) and a residual one, we must extend any deformation belonging to $\big(H^1(\mathcal{Q}_\delta)\big)^3$ in a neighborhood of the lateral boundary Γ_δ of the shell. To this end we will use Lemma 3.2 below. The proof of this lemma is identical to the one of Lemma 3.2 of [17] upon replacing the strain semi-norm of a displacement field by the norm of the distance between the gradient of a deformation v and $S0(3)$.

Lemma 3.2. *Let δ be fixed in $(0,\delta_0]$ such that $3\delta\le\eta_0$ and set*

$$\mathcal{Q}^{'}_\delta=\Phi(\omega_{3\delta}\times]-\delta,\delta[).$$

There exists an extension operator P_δ from $\big(H^1(\mathcal{Q}_\delta)\big)^3$ into $\big(H^1(\mathcal{Q}^{'}_\delta)\big)^3$ such that

$$\begin{aligned}&\forall v\in\big(H^1(\mathcal{Q}_\delta)\big)^3,\qquad P_\delta(v)\in\big(H^1(\mathcal{Q}^{'}_\delta)\big)^3,\qquad P_\delta(v)_{|\mathcal{Q}_\delta}=v,\\ &||dist\big(\nabla_x P_\delta(v),SO(3)\big)||_{L^2(\mathcal{Q}^{'}_\delta)}\le c||dist\big(\nabla_x v,SO(3)\big)||_{L^2(\mathcal{Q}_\delta)},\end{aligned}$$

with a constant c which only depends on $\partial\omega$ and on the constants appearing in inequalities (2.3).

Let us now precise the extension operator P_δ near a part of the boundary where $v=I_d$.

Let γ_0 be an open subset of $\partial\omega$ which made of a finite number of connected components (whose closure are disjoint) and v be a deformation such that $v=I_d$ on $\Gamma_{0,\delta}=\Phi(\gamma_0\times]-\delta,\delta[)$. Let $\gamma^{'}_{0,\delta}$ be the domain

$$\gamma^{'}_{0,\delta}=\big\{(s_1,s_2)\in\gamma_0\mid dist((s_1,s_2),E_0)>3\delta\big\}$$

where E_0 denotes the extremities of γ_0. We set

$$\begin{aligned}\mathcal{Q}^1_\delta&=\Phi\big(\{(s_1,s_2)\in(\omega_{3\delta}\setminus\omega)\mid dist((s_1,s_2),\gamma^{'}_{0,\delta})<3\delta\}\times]-\delta,\delta[\big),\\ \mathcal{Q}^2_\delta&=\Phi\big(\{(s_1,s_2)\in\omega_{3\delta}\mid dist((s_1,s_2),\gamma_0)<6\delta\}\times]-\delta,\delta[\big).\end{aligned}$$

Indeed, up to choosing δ_0 small enough, we can assume that $\mathcal{Q}^2_\delta$ has the same number of connected components as γ_0. The open set $\mathcal{Q}^1_\delta$ is included into $\mathcal{Q}^{'}_\delta\setminus\mathcal{Q}_\delta$. According to the construction of P_δ given in [17], we can extend the deformation v by choosing $P_\delta(v)=I_d$ in $\mathcal{Q}^1_\delta$ together with the following estimates

$$\left\{\begin{aligned}&||\nabla_x P_\delta(v)-\mathbf{I}_3||_{(L^2(\mathcal{Q}^2_\delta))^9}\le C||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)},\\ &||P_\delta(v)-I_d||_{(L^2(\mathcal{Q}^2_\delta))^3}\le C\delta||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}.\end{aligned}\right.\tag{3.2}$$

From now we assume that $3\delta\le\eta_0$ and then any deformation v belonging to $\big(H^1(\mathcal{Q}_\delta)\big)^3$ is extended to a deformation belonging to $\big(H^1(\mathcal{Q}^{'}_\delta)\big)^3$ which we still denote by v.

Now we are in a position to reproduce the technique developed in [17] in order to obtain a covering of the shell (the reader is referred to Section 3.3 of this paper for further details). Let $\mathcal{N}_\delta$ be the set of every $(k,l)\in\mathbb{Z}^2$ such that the open set

$$\omega_{\delta,(k,l)}=]k\delta,(k+1)\delta[\times]l\delta,(l+)\delta[$$

is included in $\omega_{3\delta}$ and let $\mathcal{N}_\delta^{'}$ be the set of every $(k,l)\in\mathcal{N}_\delta$ such that

$$((k+1)\delta,l\delta),\ \ (k\delta,(l+1)\delta),\ \ (k+1)\delta,(l+1)\delta)\ \text{ are in }\ \mathcal{N}_\delta.$$

We set $\Omega_{\delta,(k,l)}=\omega_{\delta,(k,l)}\times]-\delta,\delta[$.

By construction of the above covering, we have

$$\omega\subset\bigcup_{(k,l)\in\mathcal{N}_\delta^{'}}\overline{\omega}_{\delta,(k,l)}.$$

According to [17], there exist two constants R and R_1, which depend on ω and on the constants $c_0^{'}$, $c_1^{'}$, $c^{'}$ and $C^{'}$ (see(2.3)), such that for any $\delta\le(0,\eta_0/3]$ the open set $\mathcal{Q}_{\delta,(k,l)}=\Phi(\Omega_{\delta,(k,l)})$ has a diameter less than $R\delta$ and it is star-shaped with respect to a ball of radius $R_1\delta$.

As a convention and from now on, we will say that a constant C which depends only upon $\partial\omega$ and on the constants $c_0^{'}$, $c_1^{'}$, $c^{'}$ and $C^{'}$ depends on the mid-surface S and we write $C(S)$.

Since the ratio $\dfrac{R\delta}{R_1\delta}$ of each part $\mathcal{Q}_{\delta,(k,l)}$ does not depend on δ, Theorem 3.1 gives a constant $C(S)$. Let v be a deformation in $(H^1(\mathcal{Q}_\delta))^3$ extended to a deformation belonging to $(H^1(\mathcal{Q}_\delta^{'}))^3$. Applying Theorem 3.1 upon each part $\mathcal{Q}_{\delta,(k,l)}$ for $(k,l)\in\mathcal{N}_\delta$, there exist $\mathbf{R}_{\delta,(k,l)}\in SO(3)$ and $\mathbf{a}_{\delta,(k,l)}\in\mathbb{R}^3$ such that

$$\text{(3.3)}\qquad\begin{cases}||\nabla_x v-\mathbf{R}_{\delta,(k,l)}||_{(L^2(\mathcal{Q}_{\delta,(k,l)}))^{3\times3}}\le C(S)||dist(\nabla_x v;SO(3))||_{L^2(\mathcal{Q}_{\delta,(k,l)})}\\ ||v-\mathbf{a}_{\delta,(k,l)}-\mathbf{R}_{\delta,(k,l)}\big(x-\phi(k\delta,l\delta)\big)||_{(L^2(\mathcal{Q}_{\delta,(k,l)}))^3}\le C(S)R(S)\delta||dist\big(\nabla_x v;SO(3)\big)||_{L^2(\mathcal{Q}_{\delta,(k,l)})}.\end{cases}$$

For any $(k,l)\in\mathcal{N}_\delta$ such that $(k+1,l)\in\mathcal{N}_\delta$, the open set $\mathcal{Q}_{\delta,(k,l)}^{'}=\Phi(](k+1/2)\delta,(k+3/2)\delta[\times]l\delta,(l+1)\delta[\times]-\delta,\delta[)$ also have a diameter less than $R(S)\delta$ and it is also star-shaped with respect to a ball of radius $R_1(S)\delta$ (see Section 3.3 in [17]). We apply again Theorem 2.1 in the domain $\mathcal{Q}_{\delta,(k,l)}^{'}$. This gives a rotation $\mathbf{R}_{\delta,(k,l)}^{'}$. In the domain $\mathcal{Q}_{\delta,(k,l)}^{'}\cap\mathcal{Q}_{\delta,(k,l)}$ we eliminate $\nabla_x v$ in order to evaluate $|||\mathbf{R}_{\delta,(k,l)}-\mathbf{R}_{\delta,(k,l)}^{'}|||$. Then we evaluate $|||\mathbf{R}_{\delta,(k+1,l)}-\mathbf{R}_{\delta,(k,l)}^{'}|||$. Finally it leads to

$$\text{(3.4)}\quad |||\mathbf{R}_{\delta,(k+1,l)}-\mathbf{R}_{\delta,(k,l)}|||\le\frac{C(S)}{\delta^{3/2}}\big\{\|dist(\nabla_x v;SO(3))\|_{L^2(\mathcal{Q}_{\delta,(k,l)})}+||dist(\nabla_x v;SO(3))||_{L^2(\mathcal{Q}_{\delta,(k+1,l)})}\big\}.$$

In the same way, we prove that for any $(k,l)\in\mathcal{N}_\delta$ such that $(k,l+1)\in\mathcal{N}_\delta$ we have

$$\text{(3.5)}\quad |||\mathbf{R}_{\delta,(k,l+1)}-\mathbf{R}_{\delta,(k,l)}|||\le\frac{C(S)}{\delta^{3/2}}\big\{\|dist(\nabla_x v;SO(3))\|_{L^2(\mathcal{Q}_{\delta,(k,l)})}+||dist(\nabla_x v;SO(3))||_{L^2(\mathcal{Q}_{\delta,(k,l+1)})}\big\}$$

3.3. First decomposition of a deformation

In this section any deformation $v\in\big(H^1(\mathcal{Q}_\delta)\big)^3$ of the shell $\mathcal{Q}_\delta$ is decomposed as

$$\text{(3.6)}\qquad v(s)=\mathcal{V}(s_1,s_2)+s_3\mathbf{R}_a(s_1,s_2)\mathbf{n}(s_1,s_2)+\overline{v}_a(s),\qquad s\in\Omega_\delta,$$

where $\mathcal{V}$ belongs to $\big(H^1(\omega)\big)^3$, $\mathbf{R}_a$ belongs to $\big(H^1(\omega)\big)^{3\times3}$ and $\overline{v}_a$ belongs to $\big(H^1(\mathcal{Q}_\delta)\big)^3$. The map $\mathcal{V}$ is the mean value of v over the fibers while the second term $s_3\mathbf{R}_a(s_1,s_2)\mathbf{n}(s_1,s_2)$ is an approximation of the rotation of the fiber (of the shell) which contains the point $\phi(s_1,s_2)$. The sum of the two first terms $\mathcal{V}(s_1,s_2)+s_3\mathbf{R}_a(s_1,s_2)\mathbf{n}(s_1,s_2)$ is called the elementary deformation of first type of the shell.

The matrix $\mathbf{R}_a$ is defined as the Q_1 interpolate at the vertices of the cell $\omega_{\delta,(k,l)} =]k\delta,(k+1)\delta[\times]l\delta,(l+1)\delta[$ of the four elements $\mathbf{R}_{\delta,(k,l)}$, $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k,l+1)}$ and $\mathbf{R}_{\delta,(k+1,l+1)}$ belonging to $SO(3)$ (see the previous subsection). We can always define paths in $SO(3)$ from $\mathbf{R}_{\delta,(k,l)}$ to $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k,l)}$ to $\mathbf{R}_{\delta,(k,l+1)}$, $\mathbf{R}_{\delta,(k+1,l)}$ to $\mathbf{R}_{\delta,(k+1,l+1)}$ and $\mathbf{R}_{\delta,(k,l+1)}$ to $\mathbf{R}_{\delta,(k+1,l+1)}$. That gives continuous maps from the edges of the domain $\omega_{\delta,(k,l)}$ into $SO(3)$. If it is possible to extend these maps in order to obtain a continuous function from $\omega_{\delta,(k,l)}$ into $SO(3)$, then it means that the loop passing trough $\mathbf{R}_{\delta,(k,l)}$, $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k+1,l+1)}$ and $\mathbf{R}_{\delta,(k,l+1)}$ is homotopic to the constant loop equal to $\mathbf{R}_{\delta,(k,l)}$. But the fundamental group $\pi_1\big(SO(3),\mathbf{R}_{\delta,(k,l)}\big)$ is isomorphic to $\mathbb{Z}_2$ (the group of odd and even integers), hence the extension does not always exist. That is the reason why we use here a Q_1 interpolate in order to define an approximate field of rotations $\mathbf{R}_a$. In the next subsection we show that if the matrices $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k+1,l+1)}$ and $\mathbf{R}_{\delta,(k,l+1)}$ are in a neighborhood of $\mathbf{R}_{\delta,(k,l)}$ then this extension exists and we give in Theorem 3.4 a simple condition to do so.

Theorem 3.3. *Let $v \in \big(H^1(\mathcal{Q}_\delta)\big)^3$, there exist an elementary deformation (of first type) $\mathcal{V} + s_3\mathbf{R}_a\mathbf{n}$ and a deformation $\overline{v}_a$ satisfying (3.6) and such that*

$$
(3.7)\qquad
\begin{cases}
||\overline{v}_a||_{(L^2(\Omega_\delta))^3} \le C\delta||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\
||\nabla_s\overline{v}_a||_{(L^2(\Omega_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\
\Big\|\dfrac{\partial \mathbf{R}_a}{\partial s_\alpha}\Big\|_{(L^2(\omega))^9} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\
\Big\|\dfrac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{R}_a\mathbf{t}_\alpha\Big\|_{(L^2(\omega))^3} \le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\
\big\|\nabla_x v - \mathbf{R}_a\big\|_{(L^2(\Omega_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\
||dist(\mathbf{R}_a, SO(3))||_{L^2(\omega)} \le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}
\end{cases}
$$

where the constant C does not depend on δ.

Proof. The field $\mathcal{V}$ is defined by

$$
(3.8)\qquad \mathcal{V}(s_1,s_2) = \frac{1}{2\delta}\int_{-\delta}^{\delta} v(s_1,s_2,s_3)ds_3, \quad \text{a.e. in } \omega.
$$

Then we define the field $\mathbf{R}_a$ as following

$$
\forall (k,l)\in\mathcal{N}_\delta, \qquad \mathbf{R}_a(k\delta,l\delta) = \mathbf{R}_{\delta,(k,l)}
$$

and for any $(s_1,s_2)\in\omega_{\delta,(k,l)}$, $\mathbf{R}_a(s_1,s_2)$ is the Q_1 interpolate of the values of $\mathbf{R}_a$ at the vertices of the cell $\omega_{\delta,(k,l)}$.
Finally we define the field $\overline{v}_a$ by

$$
\overline{v}_a(s) = v(s) - \mathcal{V}(s_1,s_2) - s_3\mathbf{R}_a(s_1,s_2)\mathbf{n}(s_1,s_2) \qquad \text{a.e. in } \Omega_\delta.
$$

From (3.4) and (3.5) we get the third estimate in (3.7). By definition of $\mathbf{R}_a$ we obtain

$$
(3.9)\qquad \sum_{(k,l)\in\mathcal{N}'_\delta} ||\mathbf{R}_a - \mathbf{R}_{\delta,(k,l)}||^2_{(L^2(\omega_{\delta,(k,l)}))^9} \le \frac{C}{\delta}||dist(\nabla_x v; SO(3))||^2_{L^2(\mathcal{Q}_\delta)}.
$$

Taking the mean value of v on the fibers and using definition (3.8) of $\mathcal{V}$ it leads

$$
(3.10)\qquad \sum_{(k,l)\in\mathcal{N}'_\delta} ||\mathcal{V} - \mathbf{a}_{\delta,(k,l)} - \mathbf{R}_{\delta,(k,l)}\big(\phi - \phi(k\delta,l\delta)\big)||^2_{(L^2(\omega_{\delta,(k,l)}))^3} \le C\delta||dist(\nabla_x v, SO(3))||^2_{L^2(\mathcal{Q}_\delta)}.
$$

From (3.3), (3.9), (3.10) and the definition of $\overline{v}_a$ we get the first estimate in (3.7).

We compute the derivatives of the deformation v to get

$$(3.11)\qquad \frac{\partial v}{\partial s_1}=\nabla_x v\,\big(\mathbf{t}_1+s_3\frac{\partial \mathbf{n}}{\partial s_1}\big),\qquad \frac{\partial v}{\partial s_2}=\nabla_x v\,\big(\mathbf{t}_2+s_3\frac{\partial \mathbf{n}}{\partial s_2}\big),\qquad \frac{\partial v}{\partial s_3}=\nabla_x v\,\mathbf{n}.$$

We consider the restrictions of these derivatives to $\Omega_{\delta,(k,l)}$. Then, from (3.3) and (3.9) we have

$$(3.12)\qquad \Big\|\frac{\partial v}{\partial s_\alpha}-\mathbf{R}_a\big(\mathbf{t}_\alpha+s_3\frac{\partial \mathbf{n}}{\partial s_\alpha}\big)\Big\|^2_{(L^2(\Omega_\delta))^3}+\Big\|\frac{\partial v}{\partial s_3}-\mathbf{R}_a\mathbf{n}\Big\|^2_{(L^2(\Omega_\delta))^3}\le C\,||dist(\nabla_x v,SO(3))||^2_{L^2(\mathcal{Q}_\delta)}.$$

By taking the mean value of $\dfrac{\partial v}{\partial s_\alpha}-\mathbf{R}_a\big(\mathbf{t}_\alpha+s_3\dfrac{\partial \mathbf{n}}{\partial s_\alpha}\big)$ on the fibers we obtain the fourth inequality in (3.7). Observe now that

$$(3.13)\qquad \frac{\partial \overline{v}_a}{\partial s_\alpha}=\frac{\partial v}{\partial s_\alpha}-\frac{\partial \mathcal{V}}{\partial s_\alpha}-s_3\mathbf{R}_a\frac{\partial \mathbf{n}}{\partial s_\alpha}-s_3\frac{\partial \mathbf{R}_a}{\partial s_\alpha}\mathbf{n},\qquad \frac{\partial \overline{v}_a}{\partial s_3}=\frac{\partial v}{\partial s_3}-\mathbf{R}_a\mathbf{n}.$$

Then, from (3.12) and the third and fourth inequalities in (3.7) we obtain the second estimate in (3.7). The fifth inequality in (3.7) is an immediate consequence of (3.3) and (3.9). The last estimate of (3.7) is due to (3.4), (3.5) and to the very definition of the field $\mathbf{R}_a$. □

Since the matrices $\mathbf{R}_{\delta,(k,l)}$ belong to $SO(3)$, the function $\mathbf{R}_a$ is uniformly bounded and satisfies

$$||\mathbf{R}_a||_{(L^\infty(\omega))^9}\le\sqrt{3}.$$

Let (k,l) be in $\mathcal{N}_\delta$. By a straightforward computation, for any $(s_1,s_2)\in\omega_{\delta,(k,l)}$ we obtain

$$\begin{aligned}|||\mathbf{R}_a(s_1,s_2)\mathbf{R}_a^T(s_1,s_2)-\mathbf{I}_3|||\le C\big\{&|||\mathbf{R}_{\delta,(k,l)}-\mathbf{R}_{\delta,(k+1,l)}|||+|||\mathbf{R}_{\delta,(k,l)}-\mathbf{R}_{\delta,(k,l+1)}|||\\&+|||\mathbf{R}_{\delta,(k,l+1)}-\mathbf{R}_{\delta,(k+1,l+1)}|||+|||\mathbf{R}_{\delta,(k+1,l)}-\mathbf{R}_{\delta,(k+1,l+1)}|||\big\}\\ |\det\big(\mathbf{R}_a(s_1,s_2)\big)-1|\le C\big\{&|||\mathbf{R}_{\delta,(k,l)}-\mathbf{R}_{\delta,(k+1,l)}|||+|||\mathbf{R}_{\delta,(k,l)}-\mathbf{R}_{\delta,(k,l+1)}|||\\&+|||\mathbf{R}_{\delta,(k,l+1)}-\mathbf{R}_{\delta,(k+1,l+1)}|||+|||\mathbf{R}_{\delta,(k+1,l)}-\mathbf{R}_{\delta,(k+1,l+1)}|||,\big\}\end{aligned}$$

where C is an absolute constant. Hence, from (3.4) and (3.5) we deduce

$$(3.14)\qquad \left\{\begin{aligned}||\mathbf{R}_a\mathbf{R}_a^T-\mathbf{I}_3||_{(L^2(\omega))^9}&\le\frac{C}{\delta^{1/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)},\\ ||\det(\mathbf{R}_a)-1||_{L^2(\omega)}&\le\frac{C}{\delta^{1/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}.\end{aligned}\right.$$

Notice that the function $\mathbf{R}_a\mathbf{R}_a^T$ belongs to $\big(H^1(\omega)\big)^{3\times3}$ and satisfies

$$(3.15)\qquad \Big\|\frac{\partial \mathbf{R}_a\mathbf{R}_a^T}{\partial s_\alpha}\Big\|_{(L^2(\omega))^9}\le\frac{C}{\delta^{3/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}.$$

3.4. Second decomposition of a deformation.

In this section any deformation $v\in\big(H^1(\mathcal{Q}_\delta)\big)^3$ of the shell $\mathcal{Q}_\delta$ is decomposed as

$$(3.16)\qquad v(s)=\mathcal{V}(s_1,s_2)+s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2)+\overline{v}(s),\qquad s\in\Omega_\delta,$$

where $\mathcal{V}$ belongs to $\big(H^1(\omega)\big)^3$, $\mathbf{R}$ belongs to $\big(H^1(\omega)\big)^{3\times 3}$ and satisfies for a.e. $(s_1,s_2)\in\omega$: $\mathbf{R}(s_1,s_2)\in SO(3)$ and $\overline{v}$ belongs to $\big(H^1(\mathcal{Q}_\delta)\big)^3$. The first term $\mathcal{V}$ is always the mean value of v over the fibers. Now, the second one $s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2)$ describes the rotation of the fiber (of the shell) which contains the point $\phi(s_1,s_2)$. The sum of the two first terms $\mathcal{V}(s_1,s_2)+s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2)$ is called the elementary deformation of second type of the shell.

Theorem 3.4. *There exists a constant $C(S)$ (which depends only on the mid-surface S) such that for any $v\in\big(H^1(\mathcal{Q}_\delta)\big)^3$ verifying*

$$||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\le C(S)\delta^{3/2} \tag{3.17}$$

then there exist an elementary deformation of second type $\mathcal{V}+s_3\mathbf{R}\mathbf{n}$ and a deformation $\overline{v}$ satisfying (3.16) and such that

$$\begin{cases}||\overline{v}||_{(L^2(\Omega_\delta))^3}\le C\delta||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ ||\nabla_s\overline{v}||_{(L^2(\Omega_\delta))^9}\le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\mathbf{R}}{\partial s_\alpha}\Big\|_{(L^2(\omega))^9}\le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\mathcal{V}}{\partial s_\alpha}-\mathbf{R}\mathbf{t}_\alpha\Big\|_{(L^2(\omega))^3}\le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \big\|\nabla_x v-\mathbf{R}\big\|_{(L^2(\Omega_\delta))^9}\le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\end{cases} \tag{3.18}$$

where the constant C does not depend on δ.

Proof. In this proof let us denote by $C_1(S)$ the constant appearing in estimates (3.4) and (3.5). If we assume that

$$\frac{\sqrt{2}C_1(S)}{\delta^{3/2}}\|dist(\nabla_x v; SO(3))\|_{L^2(\mathcal{Q}_\delta)}\le\frac{1}{2}. \tag{3.19}$$

then, for each $(k,l)\in\mathcal{N}'_\delta$ we have using (3.4) and (3.5)

$$|||\mathbf{R}_{\delta,(k+1,l)}-\mathbf{R}_{\delta,(k,l)}|||\le\frac{1}{2},\qquad |||\mathbf{R}_{\delta,(k,l+1)}-\mathbf{R}_{\delta,(k,l)}|||\le\frac{1}{2}.$$

Thanks to Lemma A.2 in Appendix A there exists a function $\mathbf{R}\in\big(W^{1,\infty}(\omega)\big)^{3\times 3}$ such that for any $(s_1,s_2)\in\omega$ the matrix $\mathbf{R}(s_1,s_2)$ belongs to $SO(3)$ and such that

$$\forall(k,l)\in\mathcal{N}_\delta,\qquad \mathbf{R}(k\delta,l\delta)=\mathbf{R}_{\delta,(k,l)}.$$

From (3.4), (3.5) and Lemma A.2 we obtain the estimates (3.18) of the derivatives of $\mathbf{R}$. Due to the corollary of Lemma A.2 we have

$$||\mathbf{R}-\mathbf{R}_a||_{(L^2(\omega))^9}\le\frac{C}{\delta^{1/2}}||dist\big(\nabla_x v; SO(3)\big)||_{L^2(\mathcal{Q}_\delta)}. \tag{3.20}$$

All remainder estimates in (3.18) are consequences of (3.7) and (3.20). $\square$

4. Two nonlinear Korn's inequalities for shells

In this Section, we first precise the boundary conditions on the deformations and in Subsection 4.1, we deduce the first estimates on v and ∇v. Then we show that the elementary deformations of the decompositions can be imposed on the same boundary than v. The main result of Subsection 4.2 is the Korn's inequality for shells given.

Indeed these conditions depend on the boundary condition on the field v. We discuss essentially the usual case of a clamped condition on the part of the lateral boundary of $\mathcal{Q}_\delta$. Let γ_0 as in Subsection 3.2, and recall that

$$\Gamma_{0,\delta} = \Phi(\gamma_0 \times] - \delta, \delta[).$$

We assume that

$$v(x) = x \qquad \text{on} \quad \Gamma_{0,\delta}.$$

Due to the definition (3.3) of $\mathcal{V}$, we first have

$$\mathcal{V} = \phi \qquad \text{on} \quad \gamma_0. \tag{4.1}$$

4.1. First H^1- Estimates

Using the boundary condition (4.1), estimates (3.7) or (3.18) and the fact that $||\mathbf{R}_a||_{(L^\infty(\omega))^{3\times 3}} \le \sqrt{3}$ and $||\mathbf{R}||_{(L^\infty(\omega))^{3\times 3}} \le \sqrt{3}$, it leads to

$$\big\|\mathcal{V}\big\|_{(H^1(\omega))^3} \le C\Big(1 + \frac{1}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big). \tag{4.2}$$

With the help of the decompositions (3.6) or (3.16), estimates (3.7) or (3.18) and (4.2) we deduce that

$$||v||_{(L^2(\mathcal{Q}_\delta))^3} + \frac{1}{\delta}||v - \mathcal{V}||_{(L^2(\mathcal{Q}_\delta))^3} + ||\nabla_x v||_{(L^2(\mathcal{Q}_\delta))^9} \le C\Big(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big).$$

The above inequality leads to the following first "nonlinear Korn's inequality for shells":

$$||v - I_d||_{(L^2(\mathcal{Q}_\delta))^3} + ||\nabla_x v - \mathbf{I}_3||_{(L^2(\mathcal{Q}_\delta))^9} \le C\Big(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big) \tag{4.3}$$

together with

$$||(v - I_d) - (\mathcal{V} - \phi)||_{(L^2(\mathcal{Q}_\delta))^3} \le C\delta\Big(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big).$$

Let us notice that inequality (4.3) can be obtained without using the decompositions of the deformation. Indeed,we first have

$$|||\nabla v(x)||| \le dist(\nabla v(x), SO(3)) + \sqrt{3}, \qquad \text{for a.e. } x$$

so that by integration

$$||\nabla_x v||_{(L^2(\mathcal{Q}_\delta))^9} \le C\Big(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big).$$

Poincaré inequality then leads to (4.3). This is the technique used to derive estimates in [16].

4.2. Further H^1- Estimates

In this subsection, we derive a boundary condition on $\mathbf{R}_a$ and $\mathbf{R}$ on γ_0 using the extension given in Subsection 3.2. We prove the following lemma:

Lemma 4.1. *In Theorem 3.3 (respectively in Theorem 3.4), we can choose* $\mathbf{R}_a$ *(resp.* $\mathbf{R}$*) such that*

$$\mathbf{R}_a = \mathbf{I}_3 \quad on\ \gamma_0, \qquad (resp. \quad \mathbf{R} = \mathbf{I}_3 \quad on\ \gamma_0),$$

without modifications in the estimates of these theorems.

Proof. Recall that $\gamma'_{0,\delta}$, $\mathcal{Q}^1_\delta$ and $\mathcal{Q}^2_\delta$ are defined in subsection 3.2. We also set

$$\mathcal{Q}^3_\delta = \Phi\big(\{(s_1,s_2)\in\omega_{3\delta} \mid dist((s_1,s_2),\gamma_0) < 3\delta\}\times]-\delta,\delta[\big)$$

Let us consider the following function

$$\rho_\delta(s_1,s_2) = \inf\big\{1, \sup\big(0, \frac{1}{3\delta} dist((s_1,s_2),\gamma_0) - 1\big)\big\}, \qquad (s_1,s_2)\in\mathbb{R}^2.$$

This function belongs to $W^{1,\infty}(\mathbb{R}^2)$ and it is equal to 1 if $dist((s_1,s_2),\gamma_0) > 6\delta$ and to 0 if $dist((s_1,s_2),\gamma_0) < 3\delta$. Let v_δ be the deformation defined by

$$v_\delta(s) = \phi(s_1,s_2) + s_3\mathbf{n}(s_1,s_2) + \rho_\delta(s_1,s_2)\big(v(s) - \phi(s_1,s_2) - s_3\mathbf{n}(s_1,s_2)\big) \qquad \text{for a.e.} \qquad s\in\omega_{3\delta}\times]-\delta,\delta[.$$

By definition of v_δ, we have

$$v_\delta = v \qquad \text{in } \mathcal{Q}'_\delta\setminus\mathcal{Q}^2_\delta, \qquad v_\delta = I_d \qquad \text{in } \mathcal{Q}^3_\delta.$$

which gives with (3.2)

$$\begin{cases} ||\nabla_x v - \nabla_x v_\delta||_{(L^2(\mathcal{Q}'_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}, \\ \qquad ||v - v_\delta||_{(L^2(\mathcal{Q}'_\delta))^3} \le C\delta||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}. \end{cases} \tag{4.4}$$

Hence

$$\begin{cases} ||dist(\nabla_x v_\delta, SO(3))||_{L^2(\mathcal{Q}'_\delta)} \le ||\nabla_x v - \nabla_x v_\delta||_{(L^2(\mathcal{Q}'_\delta))^9} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}'_\delta)} \\ \qquad \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \end{cases} \tag{4.5}$$

where the constant does not depend on δ.

Since $v_\delta = I_d$ in $\mathcal{Q}^2_\delta$, the $\mathbf{R}_a$'s and the $\mathbf{R}$'s given by application of Theorem 3.3 or 3.4 to the deformation v_δ are both equal to $\mathbf{I}_3$ over γ_0. Estimate(3.7) and (3.18) of these theorem together with (4.4)-(4.5) show that Theorems 3.3 and 3.4 hold true for v with $\mathbf{R}_a = \mathbf{I}_3$ and $\mathbf{R} = \mathbf{I}_3$ on γ_0. □

The next theorem gives a second nonlinear Korn's inequalities, which is an improvement of (4.3) for energies of order smaller than $\delta^{3/2}$ and an estimate on $v - \mathcal{V}$ which permit to precise the scaling of the applied forces in Section 7.

Theorem 4.2. *(A second nonlinear Korn's inequality for shells) There exists a constant* C *which does not depend upon* δ *such that for all* $v \in \big(H^1(\mathcal{Q}_\delta)\big)^3$ *such that* $v = I_d$ *on* $\Gamma_{0,\delta}$

$$||v - I_d||_{(L^2(\mathcal{Q}_\delta))^3} + ||\nabla_x v - \mathbf{I}_3||_{(L^2(\mathcal{Q}_\delta))^9} \le \frac{C}{\delta}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}, \tag{4.6}$$

and

$$||(v - I_d) - (\mathcal{V} - \phi)||_{(L^2(\mathcal{Q}_\delta))^3} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}, \tag{4.7}$$

where $\mathcal{V}$ is given by (3.8).

Proof. From the decomposition (3.6), Theorem 3.3 and the boundary condition on $\mathbf{R}_a$ given by Lemma 4.1, the use of Poincaré's inequality gives

$$\begin{cases} ||\mathbf{R}_a - \mathbf{I}_3||_{(H^1(\omega))^9} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \\ \Big\|\dfrac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{t}_\alpha\Big\|_{(L^2(\omega))^3} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \end{cases} \tag{4.8}$$

Using the fact that $\mathbf{t}_\alpha = \dfrac{\partial \phi}{\partial s_\alpha}$ and the boundary condition (4.1) on $\mathcal{V}$, it leads to

$$||\mathcal{V} - \phi||_{(L^2(\omega))^3} \le \frac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta}.$$

Using again the decomposition (3.6) and Theorem 3.3, the above estimate implies that $v - I_d$ satisfies the nonlinear Korn's inequality (4.6). At last the decomposition (3.6), which implies that $(v - I_d) - (\mathcal{V} - \phi) = (\mathbf{R}_a - \mathbf{I}_3)s_3\mathbf{n} + \overline{v}_a$, the first estimate in (3.7) and (4.8) permit to obtain (4.7). □

Let us compare the two Korn's inequalities (4.3) and (4.6). Indeed they are equivalent for energies of order $\delta^{3/2}$. For energies order smaller than $\delta^{3/2}$, (4.6) is better (4.3) which is then more relevant in general for thin structures.

The decomposition technique given in Section 3 also allows to estimate the linearized strain tensor of an admissible deformation. This is the object of the lemma below.

Lemma 4.3 *There exists a constant C which does not depend upon δ such that for all $v \in \big(H^1(\mathcal{Q}_\delta)\big)^3$ such that $v = I_d$ on $\Gamma_{0,\delta}$*

$$\big\|\nabla_x v + (\nabla_x v)^T - 2\mathbf{I}_3\big\|_{(L^2(\mathcal{Q}_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big\{1 + \frac{1}{\delta^{5/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\Big\}. \tag{4.9}$$

Proof. In view of the decomposition (3.6) and Theorem 3.3 we have

$$\big\|\nabla_x v + (\nabla_x v)^T - 2\mathbf{I}_3\big\|_{(L^2(\Omega_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} + C\delta^{1/2}\big\|\mathbf{R}_a + \mathbf{R}_a^T - 2I_3\big\|_{(L^2(\omega))^9}. \tag{4.10}$$

Due to the equalities

$$\begin{aligned} \mathbf{R}_a + \mathbf{R}_a^T - 2\mathbf{I}_3 &= \mathbf{R}_a\mathbf{R}_a\mathbf{R}_a^T + \mathbf{R}_a^T - 2\mathbf{R}_a\mathbf{R}_a^T + \mathbf{R}_a(\mathbf{I}_3 - \mathbf{R}_a\mathbf{R}_a^T) + 2(\mathbf{R}_a\mathbf{R}_a^T - \mathbf{I}_3) \\ &= (\mathbf{R}_a - \mathbf{I}_3)^2\mathbf{R}_a^T + \mathbf{R}_a(\mathbf{I}_3 - \mathbf{R}_a\mathbf{R}_a^T) + 2(\mathbf{R}_a\mathbf{R}_a^T - \mathbf{I}_3) \end{aligned}$$

and to the first estimate in (3.14), it follows that

$$||\mathbf{R}_a + \mathbf{R}_a^T - 2\mathbf{I}_3||_{(L^2(\omega))^9} \le C||(\mathbf{R}_a - \mathbf{I}_3)^2||_{(L^2(\omega))^9} + \frac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}. \tag{4.11}$$

Since $||(\mathbf{R}_a-\mathbf{I}_3)^2||_{(L^2(\omega))^9}\le C||\mathbf{R}_a-\mathbf{I}_3||^2_{(L^4(\omega))^9}$ and the fact that the space $\big(H^1(\omega)\big)^{3\times 3}$ is continuously imbedded in $\big(L^4(\omega)\big)^{3\times 3}$, we deduce that

$$(4.12)\qquad ||(\mathbf{R}_a-\mathbf{I}_3)^2||_{(L^2(\omega))^9}\le \frac{C}{\delta^3}||dist(\nabla_x v, SO(3))||^2_{L^2(\mathcal{Q}_\delta)}.$$

From (4.10) , (4.11) and the previous estimate we finally get (4.9). □

Remark 4.4. In view of (3.7) and since the field $\mathbf{R}_a$ belongs to $(L^\infty(\omega))^{3\times 3}$, the function $(\mathbf{R}_a-\mathbf{I}_3)^2$ belongs to $\big(H^1(\omega)\big)^{3\times 3}$ with

$$\Big\|\frac{\partial(\mathbf{R}_a-\mathbf{I}_3)^2}{\partial s_\alpha}\Big\|_{(L^2(\omega))^9}\le \frac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}.$$

Hence, with Lemma 4.1, $||(\mathbf{R}_a-\mathbf{I}_3)^2||_{(L^2(\omega))^9}\le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}$ which gives together with (4.10)-(4.11)

$$\big\|\nabla_x v+(\nabla_x v)^T-2\mathbf{I}_3\big\|_{(L^2(\mathcal{Q}_\delta))^9}\le \frac{C}{\delta}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}.$$

Notice that the above estimate is worse than (4.9) at least as soon as the energy is smaller than $\delta^{1/2}$. □

Let us emphasize that in view of estimates (3.7)-(3.18), (4.3) and (4.9) one can distinguish two critical cases for the behavior of the quantity $||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}$ (which will be a bound from below of the elastic energy)

$$||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}=\begin{cases}O(\delta^{3/2}),\\ O(\delta^{5/2}).\end{cases}$$

Estimates (4.2)-(4.3) show that the behavior $||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\sim O(\delta^{1/2})$ also corresponds to an interesting case, but the estimates (3.7) and (4.8) show that the decompositions given in Theorems 3.3 and 3.4 are not relevant in this case which, as a consequence, must be analyzed by a different approach.

In the following we will describe the asymptotic behavior of a sequence of deformations v_δ which satisfies $||dist(\nabla_x v_\delta, SO(3))||_{L^2(\mathcal{Q}_\delta)}\sim O(\delta^{3/2})$.

5. Rescaling Ω_δ

As usual when dealing with a thin shell, we rescale Ω_δ using the operator

$$(\Pi_\delta w)(s_1,s_2,S_3)=w(s_1,s_2,s_3)\ \text{for any}\ \ s\in\Omega_\delta$$

defined for e.g. $w\in L^2(\Omega_\delta)$ for which $(\Pi_\delta w)\in L^2(\Omega)$. The estimates (3.7) on $\overline{v}_a$ transposed over Ω lead to

$$(5.1)\qquad \begin{cases}||\Pi_\delta\overline{v}_a||_{(L^2(\Omega))^3}\le C\delta^{1/2}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ ||\dfrac{\partial\Pi\overline{v}_a}{\partial s_1}||_{(L^2(\Omega))^3}\le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ ||\dfrac{\partial\Pi\overline{v}_a}{\partial s_2}||_{(L^2(\Omega))^3}\le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ ||\dfrac{\partial\Pi\overline{v}_a}{\partial S_3}||_{(L^2(\Omega))^3}\le C\delta^{1/2}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)},\end{cases}$$

and estimates (4.8) on $v - I_d$ give

$$
(5.2) \qquad \begin{cases} ||\Pi_\delta(v-I_d)||_{(L^2(\Omega))^3} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \\ ||\dfrac{\partial \Pi_\delta(v-I_d)}{\partial s_1}||_{(L^2(\Omega))^3} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \\ ||\dfrac{\partial \Pi_\delta(v-I_d)}{\partial s_2}||_{(L^2(\Omega))^3} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \\ ||\dfrac{\partial \Pi_\delta(v-I_d)}{\partial S_3}||_{(L^2(\Omega))^3} \le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}. \end{cases}
$$

6. Limit behavior of the deformation for $||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \sim \delta^{3/2}$

Let us consider a sequence of deformations v_δ of $\big(H^1(\mathcal{Q}_\delta)\big)^3$ such that

$$
(6.1) \qquad ||dist(\nabla_x v_\delta, SO(3))||_{L^2(\mathcal{Q}_\delta)} \le C\delta^{3/2}.
$$

For fixed $\delta > 0$, the deformation v_δ is decomposed as in Theorem 3.3 and the terms of this decomposition are denoted by $\mathcal{V}_\delta$, $\mathbf{R}_{a,\delta}$ and $\overline{v}_{a,\delta}$. If moreover the hypothesis (3.17) holds true for the sequence v_δ, then v_δ can be alternatively decomposed through (3.16) in terms of $\mathcal{V}_\delta$, $\mathbf{R}_\delta$ and $\overline{v}_\delta$ so that the estimates (3.18) of Theorem 3.5 are satisfied uniformly in δ.

In what follows we investigate the behavior of the sequences $\mathcal{V}_\delta$, $\mathbf{R}_{a,\delta}$ and $\overline{v}_{a,\delta}$. Indeed due to (3.20) all the result of this section can be easily transposed in terms of the sequence $\mathbf{R}_\delta$ and the details are left to the reader.

The estimates (3.7), (5.1) and (5.2) lead to the following lemma.

Lemma 6.1. *There exists a subsequence still indexed by δ such that*

$$
(6.2) \qquad \begin{cases} \mathcal{V}_\delta \longrightarrow \mathcal{V} \quad \text{strongly in} \quad \big(H^1(\omega)\big)^3, \\ \mathbf{R}_{a,\delta} \rightharpoonup \mathbf{R} \quad \text{weakly in} \quad \big(H^1(\omega)\big)^{3\times 3} \quad \text{and strongly in} \quad \big(L^2(\omega)\big)^{3\times 3}, \\ \dfrac{1}{\delta^2}\Pi_\delta \overline{v}_{a,\delta} \rightharpoonup \overline{v} \quad \text{weakly in} \quad \big(L^2(\omega; H^1(-1,1))\big)^3, \\ \dfrac{1}{\delta}\Big(\dfrac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_{a,\delta}\mathbf{t}_\alpha\Big) \rightharpoonup \mathcal{Z}_\alpha \quad \text{weakly in} \quad \big(L^2(\omega)\big)^3, \\ \dfrac{1}{\delta}\Big(\mathbf{R}_{a,\delta}^T\mathbf{R}_{a,\delta} - \mathbf{I}_3\Big) \rightharpoonup 0 \quad \text{weakly in} \quad \big(L^2(\omega)\big)^{3\times 3}, \end{cases}
$$

where $\mathbf{R}$ belongs $SO(3)$ for a.e. $(s_1, s_2) \in \omega$. We also have $\mathcal{V} \in \big(H^2(\omega)\big)^3$ and

$$
(6.3) \qquad \frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha.
$$

The boundaries conditions

$$
(6.4) \qquad \mathcal{V} = \phi, \qquad \mathbf{R} = \mathbf{I}_3 \qquad \text{on} \quad \gamma_0,
$$

hold true. Moreover, we have

$$
(6.5) \qquad \begin{cases} \Pi_\delta v_\delta \longrightarrow \mathcal{V} \quad \text{strongly in} \quad \big(H^1(\Omega)\big)^3, \\ \Pi_\delta(\nabla_x v_\delta) \longrightarrow \mathbf{R} \quad \text{strongly in} \quad \big(L^2(\Omega)\big)^9. \end{cases}
$$

Proof. The convergences (6.2) are direct consequences of Theorem 3.3 and estimate (4.8) excepted for what concerns the last convergence which will be established below. The compact imbedding of $\big(H^1(\omega)\big)^{3\times 3}$ in $\big(L^4(\omega)\big)^{3\times 3}$ and the first convergence in (6.2) permit to obtain

$$\left\{\begin{aligned} \mathbf{R}_{a,\delta} &\longrightarrow \mathbf{R} \quad \text{strongly in} \quad \big(L^4(\omega)\big)^{3\times 3},\\ \det(\mathbf{R}_{a,\delta}) &\longrightarrow \det(\mathbf{R}) \quad \text{strongly in} \quad L^{4/3}(\omega). \end{aligned}\right. \tag{6.6}$$

These convergences and estimates (3.14) prove that for a.e. $(s_1,s_2)\in\omega$: $\mathbf{R}(s_1,s_2)\in SO(3)$. The relation (6.3) and (6.4) and the convergences (6.5) are immediate consequences of Theorem 3.3 and of the above results. We now turn to the proof of the last convergence in (6.2). We first set

$$\widetilde{\mathbf{R}}_{a,\delta}(s_1,s_2)=\mathbf{R}_{a,\delta}\Big(\delta\Big[\frac{s_1}{\delta}\Big],\delta\Big[\frac{s_2}{\delta}\Big]\Big) \qquad \text{a.e. in } \omega.$$

From (3.4), (3.5) and (6.1) we have

$$||\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta}||_{(L^2(\omega))^{3\times 3}}\le C\delta. \tag{6.7}$$

From (6.6) and the above estimate, we deduce that

$$\widetilde{\mathbf{R}}_{a,\delta}\longrightarrow \mathbf{R} \quad \text{strongly in} \quad \big(L^2(\omega)\big)^{3\times 3}. \tag{6.8}$$

Now we derive the weak limit of the sequence $\dfrac{1}{\delta}(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})$. Let Φ be in $\mathcal{C}_0^\infty(\Omega)^{3\times 3}$ and set $M_\delta(\Phi)(s_1,s_2)=\displaystyle\int_{]0,1[^2}\Phi\Big(\delta\Big[\frac{s_1}{\delta}\Big]+z_1\delta,\delta\Big[\frac{s_2}{\delta}\Big]+z_2\delta\Big)dz_1dz_2$ for a.e. (s_1,s_2) in ω. We recall that (see [3])

$$\begin{aligned} \frac{1}{\delta}\big(\Phi-M_\delta(\Phi)\big)&\rightharpoonup 0 \quad \text{weakly in} \quad \big(L^2(\omega)\big)^{3\times 3}\\ M_\delta(\Phi)&\longrightarrow \Phi \quad \text{strongly in} \quad \big(L^2(\omega)\big)^{3\times 3} \end{aligned}$$

We write

$$\begin{aligned} \int_\omega \frac{1}{\delta}(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})\Phi &= \int_\omega \mathbf{R}_{a,\delta}\frac{1}{\delta}\big(\Phi-M_\delta(\Phi)\big)+\int_\omega\frac{1}{\delta}(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})M_\delta(\Phi)\\ &= \int_\omega \mathbf{R}_{a,\delta}\frac{1}{\delta}\big(\Phi-M_\delta(\Phi)\big)+\frac{1}{2}\int_\omega\Big(\frac{\partial\mathbf{R}_{a,\delta}}{\partial s_1}+\frac{\partial\mathbf{R}_{a,\delta}}{\partial s_2}\Big)M_\delta(\Phi)+K_\delta \end{aligned}$$

where $|K_\delta|\le C\delta||\nabla\mathbf{R}_{a,\delta}||_{(L^2(\omega))^{3\times 3}}||\nabla\Phi||_{(L^2(\omega))^{3\times 3}}$. In view of the properties of $M_\delta(\Phi)$ recalled above, of (6.2) and (6;4), we deduce from the above equality that

$$\frac{1}{\delta}(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})\rightharpoonup\frac{1}{2}\Big(\frac{\partial\mathbf{R}}{\partial s_1}+\frac{\partial\mathbf{R}}{\partial s_2}\Big) \quad \text{weakly in} \quad \big(L^2(\omega)\big)^{3\times 3}.$$

Now in order to prove the last convergence of (6.2), we write

$$\frac{1}{\delta}\Big(\mathbf{R}_{a,\delta}^T\mathbf{R}_{a,\delta}-\mathbf{I}_3\Big)=\frac{1}{\delta}\Big(\widetilde{\mathbf{R}}_{a,\delta}^T(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})+(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})^T\widetilde{\mathbf{R}}_{a,\delta}+(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})^T(\mathbf{R}_{a,\delta}-\widetilde{\mathbf{R}}_{a,\delta})\Big)$$

and we use estimates (3.14) and (6.7), the strong convergence (6.8) and the above weak convergence. $\square$

The following Corollary gives the limit of the Green-St Venant strain tensor of the sequence v_δ.

Corollary 6.2. *For the same subsequence as in Lemma 6.1 we have*

$$\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) \rightharpoonup (\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1} \qquad \textit{weakly in} \quad (L^1(\Omega))^9, \tag{6.9}$$

where the symmetric matrix $\mathbf{E}$ *is equal to*

$$\begin{pmatrix} S_3\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_1\cdot\mathbf{R}\mathbf{t}_1 & S_3\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \dfrac{1}{2}\big\{\mathcal{Z}_2\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_1\cdot\mathbf{R}\mathbf{t}_2\big\} & \dfrac{1}{2}\dfrac{\partial \overline{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_1 + \dfrac{1}{2}\mathcal{Z}_1\cdot\mathbf{R}\mathbf{n} \\ * & S_3\dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_2\cdot\mathbf{R}\mathbf{t}_2 & \dfrac{1}{2}\dfrac{\partial \overline{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_2 + \dfrac{1}{2}\mathcal{Z}_2\cdot\mathbf{R}\mathbf{n} \\ * & * & \dfrac{\partial \overline{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{n} \end{pmatrix} \tag{6.10}$$

and where $(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})$ *denotes the* 3×3 *matrix with first column* $\mathbf{t}_1$*, second column* $\mathbf{t}_2$ *and third column* $\mathbf{n}$ *and where* $(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T} = \big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\big)^T$.

Proof. First from estimate (3.7), equalities (3.13) and the convergences in Lemma 6.1, we obtain

$$\begin{aligned} \frac{1}{\delta}\big(\Pi_\delta(\nabla_x v_\delta) - \mathbf{R}_{a,\delta}\big)\mathbf{t}_\alpha &\rightharpoonup S_3\frac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n} + \mathcal{Z}_\alpha \quad \text{weakly in} \quad \big(L^2(\Omega)\big)^3,\\ \frac{1}{\delta}\big(\Pi_\delta(\nabla_x v_\delta) - \mathbf{R}_{a,\delta}\big)\mathbf{n} &\rightharpoonup \frac{\partial \overline{v}}{\partial S_3} \quad \text{weakly in} \quad \big(L^2(\Omega)\big)^3. \end{aligned}$$

Then thanks to the identity

$$\begin{aligned} \frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) &= \frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta - \mathbf{R}_{a,\delta})^T(\nabla_x v_\delta - \mathbf{R}_{a,\delta})\big) + \frac{1}{2\delta}\mathbf{R}_{a,\delta}^T\Pi_\delta(\nabla_x v_\delta - \mathbf{R}_{a,\delta})\\ &+ \frac{1}{2\delta}\Pi_\delta(\nabla_x v_\delta - \mathbf{R}_{a,\delta})^T\mathbf{R}_{a,\delta} + \frac{1}{2\delta}\big(\mathbf{R}_{a,\delta}^T\mathbf{R}_{a,\delta} - \mathbf{I}_3\big) \end{aligned},$$

and again to estimate (3.7) and Lemma 6.1 we deduce that

$$\left\{ \begin{aligned} \frac{1}{\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) \rightharpoonup &(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\Big(S_3\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n} + \mathcal{Z}_1\,|\,S_3\frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n} + \mathcal{Z}_2\,|\,\frac{\partial \overline{v}}{\partial S_3}\Big)^T\mathbf{R}\\ &+\mathbf{R}^T\Big(S_3\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n} + \mathcal{Z}_1\,|\,S_3\frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n} + \mathcal{Z}_2\,|\,\frac{\partial \overline{v}}{\partial S_3}\Big)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\\ &\text{weakly in} \quad (L^1(\Omega))^9. \end{aligned}\right. \tag{6.11}$$

Now remark that

$$\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 = \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1. \tag{6.12}$$

Indeed, deriving the relation $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$ with respect to s_α shows that $\mathbf{R}^T\dfrac{\partial \mathbf{R}}{\partial s_\alpha} + \dfrac{\partial \mathbf{R}^T}{\partial s_\alpha}\mathbf{R} = 0$. Hence, there exists an antisymmetric matrix $\mathbf{A}_\alpha \in L^2(\omega;\mathbb{R}^{3\times 3})$ such that $\dfrac{\partial \mathbf{R}}{\partial s_\alpha} = \mathbf{R}\mathbf{A}_\alpha$. Since there exists also a field $\mathbf{a}_\alpha$ belonging to $\big(L^2(\omega)\big)^3$ such that

$$\forall x\in\mathbb{R}^3, \qquad \mathbf{A}_\alpha\, x = \mathbf{a}_\alpha \wedge x.$$

Now we derive the equality $\dfrac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha$ with respect to s_β and we obtain

$$\frac{\partial^2 \mathcal{V}}{\partial s_\alpha\partial s_\beta} = \frac{\partial \mathbf{R}}{\partial s_\beta}\mathbf{t}_\alpha + \mathbf{R}\frac{\partial \mathbf{t}_\alpha}{\partial s_\beta} = \mathbf{R}\mathbf{A}_\beta\,\mathbf{t}_\alpha + \mathbf{R}\frac{\partial^2\phi}{\partial s_\alpha\partial s_\beta}.$$

It implies that $\mathbf{A}_1\mathbf{t}_2 = \mathbf{A}_2\mathbf{t}_1$ from which (6.12) follows. Taking into account the definition of the matrix $\mathbf{E}$, convergence (6.11) and the equality (6.12) show that (6.9) holds true. $\square$

Remark 6.3. *There exists a constant C such that*

$$\Big\|\frac{\partial \mathbf{R}}{\partial s_1}\Big\|_{(L^2(\omega))^9} + \Big\|\frac{\partial \mathbf{R}}{\partial s_\alpha}\Big\|_{(L^2(\omega))^9} \le C\Big(\Big\|\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1\Big\|_{L^2(\omega)} + \Big\|\frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2\Big\|_{L^2(\omega)} + \Big\|\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2\Big\|_{L^2(\omega)}\Big).$$

With the same notation as in the proof of Corollary 6.2, we have

$$\Big\|\frac{\partial \mathbf{R}}{\partial s_\alpha}\Big\|^2_{(L^2(\omega))^9} = \|\mathbf{A}_\alpha\|^2_{(L^2(\omega))^9} = 2||\mathbf{a}_\alpha||^2_{(L^2(\omega))^3}.$$

Recalling that $\mathbf{a}_1\wedge\mathbf{t}_2 = \mathbf{a}_2\wedge\mathbf{t}_1$, we obtain $\mathbf{a}_\alpha\cdot\mathbf{n} = 0$ and then

$$\Big\|\frac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n}\Big\|^2_{(L^2(\omega))^3} = ||\mathbf{a}_\alpha\wedge\mathbf{n}||^2_{(L^2(\omega))^3} = ||\mathbf{a}_\alpha||^2_{(L^2(\omega))^3} = \frac{1}{2}\Big\|\frac{\partial \mathbf{R}}{\partial s_\alpha}\Big\|^2_{(L^2(\omega))^9}.$$

7. Nonlinear elastic shells

In this section we consider a shell made of an elastic material. Its thickness 2δ is fixed and belongs to $]0, 2\delta_0]$. The local energy $W \;:\; \mathbf{S}_3 \longrightarrow \mathbb{R}^+$ is a continuous function of symmetric matrices which satisfies the following assumptions which are similar to those adopted in [14], [15] and [16] (the reader is also referred to [6] for general introduction to elasticity)

$$\text{(7.1)}\qquad \exists c>0 \quad \text{such that} \quad \forall E\in\mathbf{S}_3 \quad W(E)\ge c|||E|||^2,$$

$$\text{(7.2)}\qquad \forall\varepsilon>0,\quad \exists\theta>0,\quad \text{such that}\quad \forall E\in\mathbf{S}_3\quad |||E|||\le\theta \implies |W(E)-Q(E)|\le\varepsilon|||E|||^2,$$

where Q is a positive quadratic form defined on the set of 3×3 symmetric matrices. Remark that Q satisfies (7.1) with the same constant c. From (7.2) and the continuity of W we deduce that

$$\text{(7.3)}\qquad \forall C_0>0,\quad \exists C_1>0,\quad \text{such that}\quad \forall E\in\mathbf{S}_3\quad |||E|||\le C_0 \implies |W(E)|\le C_1|||E|||.$$

Still following [6], for any 3×3 matrix F, we set

$$\text{(7.4)}\qquad \widehat{W}(F) = \begin{cases} W\Big(\dfrac{1}{2}(F^TF-\mathbf{I}_3)\Big) & \text{if}\quad \det(F)>0\\ +\infty & \text{if}\quad \det(F)\le 0.\end{cases}$$

Remark that due to (7.1), (7.4) and to the inequality $|||F^TF-\mathbf{I}_3|||\ge dist(F,SO(3))$ if $\det(F)>0$, we have

$$\text{(7.5)}\qquad \widehat{W}(F)\ge\frac{c}{4}dist(F,SO(3))^2$$

for any matrix F.

Remark 7.1. As a classical example of a local elastic energy satisfying the above assumptions, we mention the following St Venant-Kirchhoff's law (see [6], [12]) for which

$$\widehat{W}(F) = \begin{cases} \dfrac{\lambda}{8}\big(tr(F^TF-\mathbf{I}_3)\big)^2 + \dfrac{\mu}{4}tr\big((F^TF-\mathbf{I}_3)^2\big) & \text{if}\quad \det(F)>0\\ +\infty \qquad \text{if} \qquad \det(F)\le 0.\end{cases}$$

In order to take into account the boundary condition on the admissible deformations we introduce the space

$$\mathbf{U}_\delta = \Big\{ v \in (H^1(\mathcal{Q}_\delta))^3 \mid v = I_d \quad \text{on} \quad \Gamma_{0,\delta} \Big\}. \tag{7.6}$$

Now we assume that the shell is submitted to applied volume forces $f_\delta \in (L^2(\Omega_\delta))^3$ and we define the total energy $J(v)$ over $\mathbf{U}_\delta$ by

$$J(v) = \int_{\mathcal{Q}_\delta} \widehat{W}(\nabla_x v)(x)dx - \int_{\mathcal{Q}_\delta} f_\delta(x) \cdot v(x)dx. \tag{7.7}$$

To introduce the scaling on f_δ, let us consider f in $(L^2(\omega))^3$ and g in $(L^2(\Omega))^3$ such that

$$\int_{-1}^{1} g(s_1, s_2, S_3)dS_3 = 0 \qquad \text{for a.e. } (s_1, s_2) \in \omega. \tag{7.8}$$

Let $\kappa \geq 1$ and assume that the force f_δ is given for $x = \Phi(s)$ by

$$f_\delta(x) = \delta^\kappa f(s_1, s_2) + \delta^{\kappa-1} g\Big(s_1, s_2, \frac{s_3}{\delta}\Big) \qquad \text{for a.e. } x \in \mathcal{Q}_\delta. \tag{7.9}$$

The fact remains that to find a minimizer or to find a deformation that approaches the minimizer of $J(v)$ or of $J(v) - J(I_d)$ is the same. Let v be in $\mathbf{U}_\delta$, thanks to (2.3), (4.6), (4.7), (7.8) and (7.9), we obtain

$$\Big| \int_{\mathcal{Q}_\delta} f_\delta(x) \cdot (v - I_d)(x)dx \Big| \leq C\delta^{\kappa-1/2}(||f_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3})||\text{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}. \tag{7.10}$$

In general, a minimizer of J does not exist on $\mathbf{U}_\delta$. In what follows, we will investigate the behavior of the functional $\dfrac{1}{\delta^{2\kappa-1}}\big(J(v) - J(I_d)\big)$ using Γ-convergence (see Remark 7.2 for a few arguments which justify the study of this quantity). Hence, we consider a deformation v of $\mathbf{U}_\delta$ such that

$$\frac{1}{\delta^{2\kappa-1}}\big(J(v) - J(I_d)\big) \leq C_1 \tag{7.11}$$

where C_1 does not depend on δ and v. Using (7.5) and (7.10) we obtain

$$C||\text{dist}(\nabla_x v, SO(3))||^2_{L^2(\mathcal{Q}_\delta)} - C\delta^{\kappa-1/2}(||f_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3})||\text{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \leq C_1\delta^{2\kappa-1}.$$

Hence, we have

$$||\text{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{\kappa-1/2} \tag{7.12}$$

where the constant C depends on the sum $||f||_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3}$ and of C_1.

In the spirit of what follows the proof of Theorem 4.1, let us notice that if one uses (4.3) to estimate the contribution of the forces in the energy, one obtains

$$||\text{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{(1+\kappa)/2}.$$

Indeed, the above estimate is the same as (7.12) for forces of order δ^2, which is, as far as we know, a critical case above which no asymptotic studies were achieved. For smaller forces, estimate (7.12) is much better and this shows that for smaller energies our technique of decomposition which lead to the second Korn's inequality given in Theorem 4.1 is more relevant in order to justify the limit models for $\kappa > 2$.

From the assumption (7.9) on the applied forces and the estimate (7.12), the results of Sections 3 and 4 permit to obtain estimates of $\mathcal{V}$, $\mathbf{R}_a$, $\overline{v}_a$ and $\nabla_x v - \mathbf{R}_a$ with respect to δ. Still for the deformation $v \in \mathbf{U}_\delta$ satisfying (7.11), we have using (7.4) and (7.5)

$$\begin{aligned}\frac{c}{4}||(\nabla_x v)^T\nabla_x v - \mathbf{I}_3||^2_{(L^2(\mathcal{Q}_\delta))^{3\times 3}} &\leq J(v) - J(I_d) + \int_{\mathcal{Q}_\delta} f_\delta \cdot (v - I_d)\\ &\leq C_1\delta^{2\kappa-1} + C\delta^{\kappa-1/2}(||f||_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3})||\mathrm{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}.\end{aligned}$$

Due to (7.12) we obtain the following estimate of the Green-St Venant's tensor:

$$\Big\|\frac{1}{2}\big\{(\nabla_x v)^T\nabla_x v - \mathbf{I}_3\big\}\Big\|_{(L^2(\mathcal{Q}_\delta))^{3\times 3}} \leq C\delta^{\kappa-1/2}. \tag{7.13}$$

We first deduce from the above inequality that $v \in (W^{1,4}(\mathcal{Q}_\delta))^3$ and moreover since $\kappa \geq 1$

$$||\nabla_x v||_{(L^4(\mathcal{Q}_\delta))^{3\times 3}} \leq C\delta^{\frac{1}{4}}. \tag{7.14}$$

Furthermore, there exists two strictly positive constants c and C which does not depend on δ such that for any $v \in \mathbf{U}_\delta$ satisfying (7.11) we have

$$-c\delta^{2\kappa-1} \leq J(v) - J(I_d) \leq C\delta^{2\kappa-1}. \tag{7.15}$$

We set

$$m_{\delta,\kappa} = \inf_{v\in\mathbf{U}_\delta}\big(J(v) - J(I_d)\big). \tag{7.16}$$

As a consequence of the inequality in (7.15) we have

$$-c \leq \frac{m_{\delta,\kappa}}{\delta^{2\kappa-1}} \leq 0. \tag{7.17}$$

We denote

$$m_\kappa = \underline{\lim}_{\delta\to 0}\frac{m_{\delta,\kappa}}{\delta^{2\kappa-1}}. \tag{7.18}$$

Remark 7.2. Under assumption (7.8) and (7.9) on the forces, the quantity $J(I_d)$ is of order $\delta^{\kappa+1}$. Hence if $\kappa > 2$ then due to (7.15) the infimum of $J(v)$ is of order $\delta^{\kappa+1}$. As a consequence for $\kappa > 2$, the relevant quantity which describes the limit behavior is $\dfrac{J(v_\delta) - J(I_d)}{\delta^{2\kappa-1}}$. For $\kappa = 2$, $J(I_d)$ and the infimum of $J(v)$ have the same order δ^3 and we derive the Γ- limit of the energy $\dfrac{J(v_\delta)}{\delta^3}$.

8. Limit model in the case $\kappa = 2$

In this section we derive the Γ-limit of the functional $\dfrac{J(v_\delta)}{\delta^3}$ which corresponds to the case $\kappa = 2$. We begin with the lim-inf condition.

Let $\big(v_\delta\big)_{0<\delta\leq\delta_0}$ be a sequence of deformations belonging to $\mathbf{U}_\delta$ and such that

$$\underline{\lim}_{\delta\to 0}\frac{J(v_\delta)}{\delta^3} < +\infty. \tag{8.1}$$

Upon extracting a subsequence (still indexed by δ) we can assume that the sequence (v_δ) satisfies the condition (7.11) (recall that $J(I_d)$ is of order δ^3). From the estimates of the previous section we obtain

$$
\begin{cases}
||\mathrm{dist}(\nabla_x v_\delta, SO(3))||_{L^2(\mathcal{Q}_\delta)} \le C\delta^{3/2},\\
\|\dfrac{1}{2}\{\nabla_x v_\delta^T \nabla_x v_\delta - \mathbf{I}_3\}\|_{(L^2(\mathcal{Q}_\delta))^{3\times 3}} \le C\delta^{3/2},\\
||\nabla_x v_\delta||_{(L^4(\mathcal{Q}_\delta))^{3\times 3}} \le C\delta^{1/4}
\end{cases}
\tag{8.2}
$$

For any fixed $\delta \in (0, \delta_0]$, the deformation v_δ is decomposed following (3.6) in such a way that Theorem 3.3 is satisfied. There exists a subsequence still indexed by δ such that (see Section 7)

$$
\begin{cases}
\mathcal{V}_\delta \longrightarrow \mathcal{V} \quad \text{strongly in} \quad \big(H^1(\omega)\big)^3\\
\mathbf{R}_{a,\delta} \rightharpoonup \mathbf{R} \quad \text{weakly in} \quad \big(H^1(\omega)\big)^{3\times 3},\\
\dfrac{1}{\delta}\Big(\dfrac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_{a,\delta}\mathbf{t}_\alpha\Big) \rightharpoonup \mathcal{Z}_\alpha \quad \text{weakly in} \quad \big(L^2(\omega)\big)^3,\\
\dfrac{1}{\delta^2}\Pi_\delta \overline{v}_{a,\delta} \rightharpoonup \overline{v} \quad \text{weakly in} \quad \big(L^2(\omega; H^1(-1,1))\big)^3
\end{cases}
\tag{8.3}
$$

where $\mathbf{R}(s_1, s_2)$ belongs to $SO(3)$ for a.e. $(s_1, s_2) \in \omega$, $\mathcal{V} \in \big(H^2(\omega)\big)^3$ together with

$$
\mathcal{V} = \phi, \qquad \mathbf{R} = \mathbf{I}_3, \quad \text{on } \gamma_0, \qquad \text{and} \quad \frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha.
\tag{8.4}
$$

Furthermore, we also have (see (6.5), (6.9) and estimate (8.2))

$$
\begin{cases}
\Pi_\delta v_\delta \longrightarrow \mathcal{V} \quad \text{strongly in} \quad \big(H^1(\Omega)\big)^3,\\
\Pi_\delta(\nabla_x v_\delta) \longrightarrow \mathbf{R} \quad \text{strongly in} \quad \big(L^2(\Omega)\big)^{3\times 3}.\\
\dfrac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3\big) \rightharpoonup (\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1} \qquad \text{weakly in} \quad (L^2(\Omega))^9,
\end{cases}
\tag{8.5}
$$

where

$$
\mathbf{E} = \begin{pmatrix}
S_3\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_1\cdot\mathbf{R}\mathbf{t}_1 & S_3\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \dfrac{1}{2}\{\mathcal{Z}_2\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_1\cdot\mathbf{R}\mathbf{t}_2\} & \dfrac{1}{2}\dfrac{\partial \overline{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_1 + \dfrac{1}{2}\mathcal{Z}_1\cdot\mathbf{R}\mathbf{n}\\
* & S_3\dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_2\cdot\mathbf{R}\mathbf{t}_2 & \dfrac{1}{2}\dfrac{\partial \overline{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_2 + \dfrac{1}{2}\mathcal{Z}_2\cdot\mathbf{R}\mathbf{n}\\
* & * & \dfrac{\partial \overline{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{n}
\end{pmatrix}
$$

Remark that, due to the decomposition (3.6), the convergences (8.3) and (8.5) imply that

$$
\frac{\Pi_\delta(v_\delta - \mathcal{V}_\delta)}{\delta} \longrightarrow S_3(\mathbf{R} - \mathbf{I}_3)\mathbf{n} \quad \text{strongly in} \quad \big(L^2(\Omega)\big)^3.
\tag{8.6}
$$

Now, recall that

$$
\begin{cases}
\quad \dfrac{J(v_\delta)}{\delta^3} = \displaystyle\int_\Omega \frac{1}{\delta^2} W\Big(\frac{1}{2}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)\Big)|\Pi_\delta \det(\nabla\Phi)|\\
-\displaystyle\int_\Omega f\cdot\Pi_\delta v_\delta|\Pi_\delta\det(\nabla\Phi)| - \int_\Omega g\cdot\frac{\Pi_\delta v_\delta}{\delta}|\Pi_\delta\det(\nabla\Phi)|
\end{cases}
\tag{8.7}
$$

In order to pass to the lim-inf in (8.7) we first recall that that $\det(\nabla\Phi)=\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})+s_3\det\Big(\dfrac{\partial\mathbf{n}}{\partial s_1}|\mathbf{t}_2\mid\mathbf{n}\Big)+s_3\det\Big(\mathbf{t}_1|\dfrac{\partial\mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)+s_3^2\det\Big(\dfrac{\partial\mathbf{n}}{\partial s_1}|\dfrac{\partial\mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)$ so that indeed $\Pi_\delta\det(\nabla\Phi)$ strongly converges to $\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})$ in $L^\infty(\Omega)$ as δ tends to 0.

We now consider the first term of the right hand side. Let $\varepsilon>0$ be fixed. Due to (7.2), there exists $\theta>0$ such that

$$\forall E\in\mathbf{S}_3,\ \ |||E|||\le\theta,\ \ W(E)\ge Q(E)-\varepsilon|||E|||^2. \tag{8.8}$$

We now use a similar argument given in [5]. Let us denote by χ_δ^θ the characteristic function of the set $A_\delta^\theta=\{s\in\Omega;|||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(s)|||\ge\theta\}$. Due to (8.2), we have

$$\mathrm{meas}(A_\delta^\theta)\le C\frac{\delta^2}{\theta^2}. \tag{8.9}$$

Using the positive character of W, (8.2) and (8.8) give

$$\begin{aligned}\int_\Omega\frac{1}{\delta^2}\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)|\Pi_\delta\det(\nabla\Phi)|&\ge\int_\Omega\frac{1}{\delta^2}W\Big(\frac12\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)\Big)(1-\chi_\delta^\theta)\Pi_\delta\det(\nabla\Phi)\\&\ge\int_\Omega Q\Big(\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(1-\chi_\delta^\theta)\Big)\Pi_\delta\det(\nabla\Phi)-C\varepsilon\end{aligned}$$

In view of (8.9), the function χ_δ^θ converges a.e. to 0 as δ tends to 0 while the weak limit of $\dfrac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(1-\chi_\delta^\theta)$ is given by (8.5). As a consequence and also using the convergence of $\Pi_\delta\det(\nabla\Phi)$ obtained above, we have

$$\varliminf_{\delta\to0}\int_\Omega\frac{1}{\delta^2}\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\det(\nabla\Phi)\ge\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})-C\varepsilon.$$

As ε is arbitrary, this gives

$$\varliminf_{\delta\to0}\int_\Omega\frac{1}{\delta^2}\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\det(\nabla\Phi)\ge\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}). \tag{8.10}$$

Using the convergences (8.5) and (8.6), it follows that

$$\lim_{\delta\to0}\Big(\int_\Omega f\cdot\Pi_\delta(v_\delta)\Pi_\delta\det(\nabla\Phi)+\int_\Omega g\cdot\frac{\Pi_\delta(v_\delta)}{\delta}\Pi_\delta\det(\nabla\Phi)\Big)=\mathbb{L}(\mathcal{V},\mathbf{R})$$

where $\mathbb{L}$ is defined by

$$\begin{cases}\mathbb{L}(\mathcal{V},\mathbf{R})=\displaystyle\int_\omega\Big(2f\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})+\int_{-1}^1 g(.,.,S_3)S_3\Big[\det\Big(\frac{\partial\mathbf{n}}{\partial s_1}|\mathbf{t}_2\mid\mathbf{n}\Big)+\det\Big(\mathbf{t}_1|\frac{\partial\mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)\Big]dS_3\Big)\cdot\mathcal{V}\\ \qquad\displaystyle+\int_\omega\Big(\int_{-1}^1 g(.,.,S_3)S_3dS_3\Big)\cdot\mathbf{R}\,\mathbf{n}\,\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}).\end{cases} \tag{8.11}$$

From (8.7), (8.10) and the above limit, we conclude that

$$\varliminf_{\delta\to0}\frac{J(v_\delta)}{\delta^3}\ge\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})-\mathbb{L}(\mathcal{V},\mathbf{R}). \tag{8.12}$$

In order to bound from below the right hand side of (8.12), we first write

$$\mathbf{E} = \begin{pmatrix} S_3 \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_{11} & S_3 \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_{12} & \dfrac{1}{2}\dfrac{\partial \overline{\overline{v}}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_1 \\ * & S_3 \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_{22} & \dfrac{1}{2}\dfrac{\partial \overline{\overline{v}}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_2 \\ * & * & \dfrac{\partial \overline{\overline{v}}}{\partial S_3}\cdot\mathbf{R}\mathbf{n} \end{pmatrix}$$

where $\overline{\overline{v}}$ is defined by

$$\overline{\overline{v}}\cdot\mathbf{R}\mathbf{t}_\alpha = \overline{v}\cdot\mathbf{R}\mathbf{t}_\alpha + S_3\mathcal{Z}_\alpha\cdot\mathbf{R}\mathbf{n} \qquad \overline{\overline{v}}\cdot\mathbf{R}\mathbf{n} = \overline{v}\cdot\mathbf{R}\mathbf{n}$$

and where

$$\mathcal{Z}_{\alpha\beta} = \frac{1}{2}\big\{\mathcal{Z}_\alpha\cdot\mathbf{R}\mathbf{t}_\beta + \mathcal{Z}_\beta\cdot\mathbf{R}\mathbf{t}_\alpha\big\}.$$

Now for almost any fixed (s_1, s_2) in ω, we apply Lemma B.1 of the appendix to the quadratic form

$$\int_{-1}^{1} Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})dS_3 = \int_{-1}^{1}\mathbf{A}\begin{pmatrix} S_3\mathbf{a}_1+\mathbf{b}_1 \\ S_3\mathbf{a}_2+\mathbf{b}_2 \\ S_3\mathbf{a}_3+\mathbf{b}_3 \\ \mathbf{c}_1(S_3) \\ \mathbf{c}_2(S_3) \\ \mathbf{c}_3(S_3)\end{pmatrix}\cdot\begin{pmatrix} S_3\mathbf{a}_1+\mathbf{b}_1 \\ S_3\mathbf{a}_2+\mathbf{b}_2 \\ S_3\mathbf{a}_3+\mathbf{b}_3 \\ \mathbf{c}_1(S_3) \\ \mathbf{c}_2(S_3) \\ \mathbf{c}_3(S_3)\end{pmatrix} dS_3$$

where $\mathbf{A}$ is a symmetric positive definite constant 6×6 matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \vdots & \mathbf{A}_2 \\ \cdots & & \cdots \\ \mathbf{A}_2^T & \vdots & \mathbf{A}_3 \end{pmatrix}$$

and where $\mathbf{a} = \big(\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1, \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2, \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2\big)$, $\mathbf{b} = \big(\mathcal{Z}_{11}, \mathcal{Z}_{12}, \mathcal{Z}_{22}\big)$ and $\mathbf{c} = \big(\dfrac{1}{2}\dfrac{\partial \overline{\overline{v}}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_1, \dfrac{1}{2}\dfrac{\partial \overline{\overline{v}}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_2, \dfrac{\partial \overline{\overline{v}}}{\partial S_3}\cdot\mathbf{R}\mathbf{n}\big)$. Then, if we define the function $\overline{w}\in(L^2(\omega; H^1(-1,1)))^3$ by

$$\overline{w} = \overline{\overline{v}} + S_3\mathbf{A}_3^{-1}\mathbf{A}_2^T(\mathcal{Z}_{11}\mathbf{e}_1 + \mathcal{Z}_{12}\mathbf{e}_2 + \mathcal{Z}_{22}\mathbf{e}_3) \tag{8.13}$$

we have

$$\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) \geq \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})$$

where

$$\widehat{\mathbf{E}} = \begin{pmatrix} S_3 \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 & S_3 \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 & \dfrac{1}{2}\dfrac{\partial \overline{w}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_1 \\ * & S_3 \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 & \dfrac{1}{2}\dfrac{\partial \overline{w}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_2 \\ * & * & \dfrac{\partial \overline{w}}{\partial S_3}\cdot\mathbf{R}\mathbf{n} \end{pmatrix} \tag{8.14}$$

Moreover since $\overline{w}$ is defined up to a function of (s_1, s_2), we can assume that $\displaystyle\int_{-1}^{1}\overline{w}(s_1, s_2, S_3)dS_3 = 0$ for almost any (s_1, s_2) in ω.

Let $\mathbf{U}_{nlin}$ be the set

$$\mathbf{U}_{nlin} = \Big\{ \big(\mathcal{V}^{'}, \mathbf{R}^{'}, \overline{w}^{'}\big) \in (H^2(\omega))^3 \times (H^1(\omega))^{3\times 3} \times (L^2(\omega; H^1((-1,1))))^3 \ \ | \\
\mathcal{V}^{'} = \phi, \quad \mathbf{R}^{'} = \mathbf{I}_3, \ \text{ on } \gamma_0, \quad \int_{-1}^{1} \overline{w}^{'}(s_1, s_2, S_3) dS_3 = 0 \quad \text{for a.e. } (s_1, s_2) \in \omega, \\
\mathbf{R}^{'}(s_1, s_2) \in SO(3) \ \text{ for a.e. } (s_1, s_2) \in \omega, \quad \frac{\partial \mathcal{V}^{'}}{\partial s_\alpha} = \mathbf{R}^{'}\mathbf{t}_\alpha \Big\}.$$

The set $\mathbf{U}_{nlin}$ is closed in the product space. For any $\big(\mathcal{V}^{'}, \mathbf{R}^{'}, \overline{v}^{'}\big) \in \mathbf{U}_{nlin}$, we denote by $\mathcal{J}_{NL}$ the following limit energy

$$\mathcal{J}_{NL}\big(\mathcal{V}^{'}, \mathbf{R}^{'}, \overline{w}^{'}\big) = \int_\Omega Q\Big((\mathbf{t}_1 \,|\, \mathbf{t}_2 \,|\, \mathbf{n})^{-T} \widehat{\mathbf{E}}^{'} (\mathbf{t}_1 \,|\, \mathbf{t}_2 \,|\, \mathbf{n})^{-1} \Big) \det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathbb{L}(\mathcal{V}^{'}, \mathbf{R}^{'}) \tag{8.15}$$

where

$$\widehat{\mathbf{E}}^{'} = \begin{pmatrix} S_3 \dfrac{\partial \mathbf{R}^{'}}{\partial s_1}\mathbf{n} \cdot \mathbf{R}^{'}\mathbf{t}_1 & S_3 \dfrac{\partial \mathbf{R}^{'}}{\partial s_1}\mathbf{n} \cdot \mathbf{R}^{'}\mathbf{t}_2 & \dfrac{1}{2} \dfrac{\partial \overline{w}^{'}}{\partial S_3} \cdot \mathbf{R}^{'}\mathbf{t}_1 \\ * & S_3 \dfrac{\partial \mathbf{R}^{'}}{\partial s_2}\mathbf{n} \cdot \mathbf{R}^{'}\mathbf{t}_2 & \dfrac{1}{2} \dfrac{\partial \overline{w}^{'}}{\partial S_3} \cdot \mathbf{R}^{'}\mathbf{t}_2 \\ * & * & \dfrac{\partial \overline{w}^{'}}{\partial S_3} \cdot \mathbf{R}^{'}\mathbf{n} \end{pmatrix} \tag{8.16}$$

Indeed notice that in $\mathbf{U}_{nlin}$ and $\mathcal{J}_{NL}$, the matrix $\mathbf{R}^{'}$ could be eliminated using the relation $\dfrac{\partial \mathcal{V}^{'}}{\partial s_\alpha} = \mathbf{R}^{'}\mathbf{t}_\alpha$ and the fact that $\mathbf{R}^{'} \in SO(3)$. Doing such a elimination would lead to an intricate expression of the strain tensor $\widehat{\mathbf{E}}^{'}$ and this is why we prefer to work with the two fields $\mathcal{V}^{'}$ and $\mathbf{R}^{'}$ and to keep the constraint in the definition of $\mathbf{U}_{nlin}$.

From (8.12)-(8.15) we deduce that

$$\mathcal{J}_{NL}\big(\mathcal{V}, \mathbf{R}, \overline{w}\big) \le \varliminf_{\delta \to 0} \frac{J(v_\delta)}{\delta^3}, \tag{8.17}$$

which gives the lim-inf condition in the definition of the Γ- convergence.

In order to obtain the lim-sup condition for the Γ-limit, we first prove the following Lemma.

Lemma 8.1. *Let $\big(\mathcal{V}, \mathbf{R}, \overline{w}\big)$ be in $\mathbf{U}_{nlin}$, there exists a sequence $\Big(\big(\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{w}_\delta\big)\Big)_{\delta > 0}$ of $(W^{2,\infty}(\omega))^3 \times (W^{1,\infty}(\omega))^{3\times 3} \times (W^{1,\infty}(\Omega))^3$ such that*

$$\mathcal{V}_\delta = \phi, \quad \mathbf{R}_\delta = \mathbf{I}_3 \quad \text{on } \gamma_0, \qquad \overline{w}_\delta = 0, \quad \text{on } \gamma_0 \times]-1, 1[, \tag{8.18}$$

with

$$\begin{cases} \mathcal{V}_\delta \longrightarrow \mathcal{V} \quad \text{strongly in} \quad (H^2(\omega))^3 \\ \mathbf{R}_\delta \longrightarrow \mathbf{R} \quad \text{strongly in} \quad (H^1(\omega))^{3\times 3} \\ \dfrac{1}{\delta}(\mathbf{R}_\delta - \mathbf{R}) \longrightarrow 0 \quad \text{strongly in} \quad (L^2(\omega))^{3\times 3} \\ \dfrac{1}{\delta}\Big(\dfrac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta \mathbf{t}_\alpha \Big) \longrightarrow 0 \quad \text{strongly in} \quad (L^2(\omega))^3 \\ \overline{w}_\delta \longrightarrow \overline{w} \quad \text{strongly in} \quad (L^2(\omega; H^1((-1,1)))^3, \\ \delta \dfrac{\partial \overline{w}_\delta}{\partial s_\alpha} \longrightarrow 0 \quad \text{strongly in} \quad (L^2(\Omega))^3, \end{cases} \tag{8.19}$$

and moreover

$$
(8.20)\qquad \begin{cases} ||dist\big(\mathbf{R}_\delta, SO(3)\big)||_{L^\infty(\omega)} \le \dfrac{1}{8}, \qquad \Big\|\dfrac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta \mathbf{t}_\alpha\Big\|_{(L^\infty(\omega))^3} \le \dfrac{1}{8},\\ ||\mathbf{R}_\delta||^2_{(W^{1,\infty}(\omega))^{3\times 3}} + ||\overline{w}_\delta||^2_{((W^{1,\infty}(\Omega))^3} \le \dfrac{1}{(4c_1^{'}\delta)^2}. \end{cases}
$$

The constant $c_1^{'}$ is given by (2.3).

Proof. For $h>0$ small enough, consider a $\mathcal{C}^\infty_0(\mathbb{R}^2)$-function ψ_h such that $0\le\psi_h\le 1$

$$
\begin{cases} \psi_h(s_1,s_2) = 1 \text{ if } dist\big((s_1,s_2),\gamma_0\big)\le h\\ \psi_h(s_1,s_2) = 0 \text{ if } dist\big((s_1,s_2),\gamma_0\big)\ge 2h. \end{cases}
$$

Indeed we can assume that

$$
(8.21)\qquad ||\psi_h||_{W^{1,\infty}(\mathbb{R}^2)} \le \frac{C}{h}, \qquad ||\psi_h||_{W^{2,\infty}(\mathbb{R}^2)} \le \frac{C}{h^2}.
$$

Since ω is bounded with a Lipschitz boundary, we first extend the fields $\mathcal{V}$ and $\mathbf{R_n}=\mathbf{Rn}$ into two fields of $(H^2(\mathbb{R}^2))^3$ and $(H^1(\mathbb{R}^2))^3$ (and we use the same notations for these extentions). We define the 3×3 matrix field $\mathbf{R}^{'}\in (H^1(\mathbb{R}^2))^{3\times 3}$ by the formula

$$
(8.22)\qquad \mathbf{R}^{'} = \Big(\frac{\partial \mathcal{V}}{\partial s_1}|\frac{\partial \mathcal{V}}{\partial s_2}|\mathbf{R_n}\Big)\Big(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}\Big)^{-1}.
$$

By construction we have $\dfrac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}^{'}\mathbf{t}_\alpha$ in $\mathbb{R}^2$ and $\mathbf{R}^{'}=\mathbf{R}$ in ω. At least, we introduce below the approximations $\mathcal{V}_h$ and $\mathbf{R}_h$ of $\mathcal{V}$ and $\mathbf{R}$ as restrictions to $\overline{\omega}$ of the following fields defined into $\mathbb{R}^2$:

$$
(8.23)\qquad \begin{cases} \mathcal{V}^{'}_h(s_1,s_2) = \dfrac{1}{9\pi h^2}\displaystyle\int_{B(0,3h)} \mathcal{V}(s_1+t_1,s_2+t_2)dt_1dt_2,\\ \mathbf{R}^{'}_h(s_1,s_2) = \dfrac{1}{9\pi h^2}\displaystyle\int_{B(0,3h)} \mathbf{R}^{'}(s_1+t_1,s_2+t_2)dt_1dt_2, \end{cases} \quad \text{a.e. } (s_1,s_2)\in\mathbb{R}^2
$$

and

$$
(8.24)\qquad \mathcal{V}_h = \phi\psi_h + \mathcal{V}^{'}_h(1-\psi_h), \qquad \mathbf{R}_h = \mathbf{I}_3\psi_h + \mathbf{R}^{'}_h(1-\psi_h), \quad \text{in } \omega.
$$

Notice that we have

$$
(8.25)\qquad \begin{aligned} &\mathcal{V}^{'}_h \in (W^{2,\infty}(\mathbb{R}^2))^3, \qquad \mathbf{R}^{'}_h \in (W^{1,\infty}(\mathbb{R}^2))^{3\times 3},\\ &\mathcal{V}_h \in (W^{2,\infty}(\omega))^3, \qquad \mathbf{R}_h \in (W^{1,\infty}(\omega))^{3\times 3}, \qquad \mathcal{V}_h=\phi, \quad \mathbf{R}_h = \mathbf{I}_3 \text{ on } \gamma_0. \end{aligned}
$$

Due to the definition (8.22) of $\mathbf{R}^{'}$ and in view of (8.23) we have

$$
(8.26)\qquad \begin{cases} \mathcal{V}^{'}_h \longrightarrow \mathcal{V} \quad \text{strongly in } (H^2(\mathbb{R}^2))^3,\\ \mathbf{R}^{'}_h \longrightarrow \mathbf{R}^{'} \quad \text{strongly in } (H^1(\mathbb{R}^2))^{3\times 3} \end{cases}
$$

and thus using estimates (8.21)

$$
\begin{cases} \mathcal{V}_h \longrightarrow \mathcal{V} & \text{strongly in } (H^2(\omega))^3, \\ \mathbf{R}_h \longrightarrow \mathbf{R} & \text{strongly in } (H^1(\omega))^{3\times 3} \end{cases} \tag{8.27}
$$

Moreover using again (8.23) and the fact that $\mathbf{R}^{'} - \mathbf{R}_h$ strongly converges to 0 in $(H^1(\mathbb{R}^2))^{3\times 3}$ we deduce that

$$
\frac{1}{h}\big(\mathbf{R}^{'}_h - \mathbf{R}^{'}\big) \longrightarrow 0 \quad \text{strongly in } (L^2(\mathbb{R}^2))^{3\times 3}
$$

and then together with (8.21), (8.22), (8.24) and (8.27) we get

$$
\begin{aligned}
\frac{1}{h}\big(\mathbf{R}_h - \mathbf{R}\big) &\longrightarrow 0 \quad \text{strongly in } (L^2(\omega))^{3\times 3},\\
\frac{1}{h}\big(\frac{\partial \mathcal{V}_h}{\partial s_\alpha} - \mathbf{R}_h\mathbf{t}_\alpha\big) &\longrightarrow 0 \quad \text{strongly in } (L^2(\omega))^{3}.
\end{aligned}
$$

We now turn to the estimate of the distance between $\mathbf{R}_h(s_1,s_2)$ and $SO(3)$ for a.e. $(s_1,s_2)\in\omega$. We apply the Poincaré-Wirtinger's inequality to the function $(u_1,u_2)\longmapsto \mathbf{R}^{'}(u_1,u_2)$ in the ball $B((s_1,s_2),3h)$. We obtain

$$
\int_{B((s_1,s_2),3h)} |||\mathbf{R}^{'}(u_1,u_2) - \mathbf{R}^{'}_h(s_1,s_2)|||^2 du_1du_2 \le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(B((s_1,s_2),3h)))^3}
$$

where C is the Poincaré-Wirtinger's constant for a ball. Since the open set ω is boundy with a Lipschitz boundary, there exists a positive constant $c(\omega)$, which depends only on ω, such that

$$
|\big(B((s_1,s_2),3h)\setminus B((s_1,s_2),2h)\big)\cap\omega| \ge c(\omega)h^2.
$$

Setting $m_h(s_1,s_2)$ the essential infimum of the function $(u_1,u_2)\mapsto |||\mathbf{R}(u_1,u_2) - \mathbf{R}^{'}_h(s_1,s_2)|||$ into the set $\big(B((s_1,s_2),3h)\setminus B((s_1,s_2),2h)\big)\cap\omega$, we then obtain

$$
c(\omega)h^2 m_h(s_1,s_2)^2 \le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(B((s_1,s_2),3h)))^3}
$$

Hence, thanks to the strong convergence of $\mathbf{R}^{'}_h$ given by (8.26), the above inequality shows that there exists $h^{'}_0$ which does not depend on $(s_1,s_2)\in\overline{\omega}$ such that for any $h\le h^{'}_0$

$$
dist(\mathbf{R}^{'}_h(s_1,s_2), SO(3)) \le 1/8 \qquad \text{for any } (s_1,s_2)\in\overline{\omega}.
$$

Now,

• in the case $dist\big((s_1,s_2),\gamma_0\big) > 2h$, $(s_1,s_2)\in\omega$, by definition of $\mathbf{R}_h$ and thanks to the above inequality we have $dist(\mathbf{R}_h(s_1,s_2), SO(3)) \le 1/8$,

• in the case $dist\big((s_1,s_2),\gamma_0\big) < h$, $(s_1,s_2)\in\omega$, by definition of $\mathbf{R}_h$ we have $\mathbf{R}_h(s_1,s_2)=\mathbf{I}_3$ and then $dist(\mathbf{R}_h(s_1,s_2), SO(3)) = 0$,

• in the case $h\le dist\big((s_1,s_2),\gamma_0\big)\le 2h$, $(s_1,s_2)\in\omega$, due to the fact that $\mathbf{R}^{'}=\mathbf{I}_3$ onto γ_0, firstly we have

$$
||\mathbf{R}^{'} - \mathbf{I}_3||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}} \le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}
$$

where $\omega_{kh,\gamma_0} = \{(s_1,s_2)\in\mathbb{R}^2 \mid dist\big((s_1,s_2),\gamma_0\big)\le kh\}$, $k\in\mathbb{N}^*$. Hence

$$
||\mathbf{R}^{'}_h - \mathbf{I}_3||^2_{(L^2(\omega_{3h,\gamma_0}))^{3\times 3}} \le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}.
$$

The constants depend only on $\partial\omega$.

Secondly, we set M_h the maximum of the function $(u_1,u_2)\longmapsto |||\mathbf{I}_3-\mathbf{R}^{'}_h(u_1,u_2)|||$ into the closed set $\big\{(u_1,u_2)\in\omega\mid h\le dist\big((u_1,u_2),\gamma_0\big)\le 2h\big\}$, and let (s_1,s_2) be in this closed subset of ω such that

$$M_h=|||\mathbf{I}_3-\mathbf{R}^{'}_h(s_1,s_2)|||.$$

Applying the Poincaré-Wirtinger's inequality in the ball $B\big((s_1,s_2),4h\big)$ we deduce that

$$\forall (s^{'}_1,s^{'}_2)\in B\big((s_1,s_2),h\big),\qquad |||\mathbf{R}^{'}_h(s^{'}_1,s^{'}_2)-\mathbf{R}^{'}_h(s_1,s_2)|||\le C||\nabla\mathbf{R}^{'}||_{(L^2(B((s_1,s_2),4h)))^3}.$$

The constant depends only on the Poincaré-Wirtinger's constant for a ball.
If M_h is larger than $C||\nabla\mathbf{R}^{'}||_{(L^2(B((s_1,s_2),4h)))^3}$ we have

$$\begin{aligned}&\pi h^2\big(M_h-C||\nabla\mathbf{R}^{'}||_{(L^2(B((s_1,s_2),4h)))^3}\big)^2\le ||\mathbf{R}^{'}_h-\mathbf{I}_3|||^2_{(L^2(B((s_1,s_2),h)))^3}\\ \le&||\mathbf{R}^{'}_h-\mathbf{I}_3||^2_{(L^2(\omega_{3h,\gamma_0}))^{3\times 3}}\le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}\end{aligned}$$

then, in all the cases we obtain

$$M_h\le C||\nabla\mathbf{R}^{'}||_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}.$$

The constant does not depend on h and $\mathbf{R}^{'}$. The above inequalities show that there exists $h^{''}_0$ such that for any $h\le h^{''}_0$

$$|||\mathbf{R}^{'}_h(s_1,s_2)-\mathbf{I}_3|||\le C||\nabla\mathbf{R}^{'}||_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}\le 1/8\qquad\text{for any } (s_1,s_2)\in\omega\quad\text{such that}\quad h\le dist\big((s_1,s_2),\gamma_0\big)\le 2h.$$

By definition of $\mathbf{R}_h$, that gives $|||\mathbf{R}_h(s_1,s_2)-\mathbf{I}_3|||\le 1/8$.

Finally, for any $h\le\max(h^{'}_0,h^{''}_0)$ and for any $(s_1,s_2)\in\overline{\omega}$ we have

$$dist(\mathbf{R}_h(s_1,s_2),SO(3))\le 1/8.$$

Using (8.22) and (8.23) we obtain (recall that $\big\|\cdot\big\|_2$ is the euclidian norm in $\mathbb{R}^3$)

$$\forall (s_1,s_2)\in\omega,\qquad \Big\|\frac{\partial\mathcal{V}^{'}_h}{\partial s_\alpha}(s_1,s_2)-\mathbf{R}^{'}_h(s_1,s_2)\mathbf{t}_\alpha(s_1,s_2)\Big\|_2\le Ch||\phi||_{(W^{2,\infty}(\omega))^3}+C\big(||\mathcal{V}||_{(H^2(\omega_{3h}))^3}+||\mathbf{R}^{'}||_{(H^1(\omega_{3h}))^{3\times 3}}\big)$$

where $\omega_{3h}=\{(s_1,s_2)\in\mathbb{R}^2\mid dist\big((s_1,s_2),\partial\omega\big)\le 3h\}$.
We have

$$\frac{\partial\mathcal{V}_h}{\partial s_\alpha}-\mathbf{R}_h\mathbf{t}_\alpha=(1-\psi_h)\Big(\frac{\partial\mathcal{V}^{'}_h}{\partial s_\alpha}-\mathbf{R}^{'}_h\mathbf{t}_\alpha\Big)+\frac{\partial\psi_h}{\partial s_\alpha}\big(\phi-\mathcal{V}^{'}_h\big).$$

Thanks to the above inequality, (8.21) and again the estimate of $|||\mathbf{R}^{'}_h-\mathbf{I}_3|||$ in the edge strip $h\le dist\big((s_1,s_2),\gamma_0\big)\le 2h$ we obtain for all $(s_1,s_2)\in\omega$

$$\begin{aligned}&\Big\|\frac{\partial\mathcal{V}_h}{\partial s_\alpha}(s_1,s_2)-\mathbf{R}_h(s_1,s_2)\mathbf{t}_\alpha(s_1,s_2)\Big\|_2\\ \le&C\big(h||\phi||_{(W^{2,\infty}(\omega))^3}+||\mathcal{V}||_{(H^2(\omega_{3h}))^3}+||\mathbf{R}^{'}||_{(H^1(\omega_{3h}))^{3\times 3}}+||\phi-\mathcal{V}||_{(H^2(\omega_{5h,\gamma_0}))^{3\times 3}}\big).\end{aligned}$$

The same argument as above imply that there exists $h_0 \le \max(h_0^{'}, h_0^{''})$ such that for any $0 < h \le h_0$ and for any $(s_1, s_2) \in \omega$ we have

$$\Big\| \frac{\partial \mathcal{V}_h}{\partial s_\alpha}(s_1, s_2) - \mathbf{R}_h(s_1, s_2)\mathbf{t}_\alpha(s_1, s_2) \Big\|_2 \le \frac{1}{8}. \tag{8.28}$$

From (8.21), (8.22), (8.23) and (8.24) there exists a positive constant C which does not depend on h such that

$$||\mathbf{R}_h||_{(W^{1,\infty}(\omega))^{3\times 3}} \le \frac{C}{h} \big\{ ||\mathcal{V}||_{(H^2(\omega))^3} + ||\mathbf{R}||_{(H^1(\omega))^{3\times 3}} \big\}. \tag{8.29}$$

Now we can choose h in term of δ. We set

$$h = \kappa\delta, \qquad \delta \in (0, \delta_0]$$

and we fixed κ in order to have $h \le h_0$ and to obtain the right hand side in (8.29) less than $\dfrac{1}{4\sqrt{2}c_1^{'}\delta}$ ($c_1^{'}$ is given by (2.3)). It is well-known that there exists a sequence $\big(\overline{w}_\delta\big)_{\delta \in (0,\delta_0]}$ satisfying (8.20), the convergences in (8.18) and the estimate

$$||\overline{w}_\delta||_{(W^{1,\infty}(\Omega))^3} \le \frac{1}{4\sqrt{2}c_1^{'}\delta}.$$

□

Let us now consider an arbitrary element $\big(\mathcal{V}, \mathbf{R}, \overline{w}\big)$ of $\mathbf{U}_{nlin}$ and the corresponding sequences $\Big(\big(\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{w}_\delta\big)\Big)_{\delta > 0}$ given by Lemma 8.1. The deformation v_δ is then defined by

$$v_\delta(s) = \mathcal{V}_\delta(s_1, s_2) + s_3\mathbf{R}_\delta(s_1, s_2)\mathbf{n}(s_1, s_2) + \delta^2 \overline{w}_\delta\Big(s_1, s_2, \frac{s_3}{\delta}\Big), \quad \text{for } s \in \Omega_\delta. \tag{8.30}$$

Step 1. Estimate on $||\Pi_\delta\big(\nabla_x v_\delta - \mathbf{R}_\delta\big)||_{L^\infty(\Omega))^{3\times 3}}$ and $||dist\big(\nabla_x v_\delta, SO(3)\big)||_{L^\infty(\omega)}$.

Using (3.11) and (8.30), we first have

$$\begin{cases} (\nabla_x v_\delta - \mathbf{R}_\delta)\mathbf{t}_\alpha = \dfrac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta \mathbf{t}_\alpha + s_3 \dfrac{\partial \mathbf{R}_\delta}{\partial s_\alpha}\mathbf{n} + \delta^2 \dfrac{\partial \overline{w}_\delta}{\partial s_\alpha} - (\nabla_x v_\delta - \mathbf{R}_\delta)s_3 \dfrac{\partial \mathbf{n}}{\partial s_\alpha} \\ (\nabla_x v_\delta - \mathbf{R}_\delta)\mathbf{n} = \delta \dfrac{\partial \overline{w}_\delta}{\partial S_3}. \end{cases} \tag{8.31}$$

We first estimate the L^∞-norm of $\Pi_\delta(\nabla_x v_\delta - \mathbf{R}_\delta)$. We have

$$\begin{aligned} &\Pi_\delta(\nabla_x v_\delta - \mathbf{R}_\delta) \cdot \Pi_\delta(\nabla_s \Phi) \\ =&\Big(\frac{\partial \mathcal{V}_\delta}{\partial s_1} - \mathbf{R}_\delta \mathbf{t}_1 + S_3\delta \frac{\partial \mathbf{R}_\delta}{\partial s_\alpha}\mathbf{n} + \delta^2 \frac{\partial \overline{w}_\delta}{\partial s_\alpha} \mid \frac{\partial \mathcal{V}_\delta}{\partial s_2} - \mathbf{R}_\delta \mathbf{t}_2 + S_3\delta \frac{\partial \mathbf{R}_\delta}{\partial s_\alpha}\mathbf{n} + \delta^2 \frac{\partial \overline{w}_\delta}{\partial s_\alpha} \mid \delta \frac{\partial \overline{w}_\delta}{\partial S_3}\Big). \end{aligned} \tag{8.32}$$

Thanks to (2.3), (8.20) and (8.28) we obtain

$$||\Pi_\delta\big(\nabla_x v_\delta - \mathbf{R}_\delta\big)||_{(L^\infty(\Omega))^{3\times 3}} \le \frac{1}{4}. \tag{8.33}$$

From (8.20) and (8.33) we deduce that there exists a positive constant C_0 such that

$$||\Pi_\delta\big((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3\big)||_{(L^\infty(\Omega))^{3\times 3}} \le C_0. \tag{8.34}$$

In view of (8.18) and (8.33) we deduce that

$$||dist\big(\nabla_x v_\delta, SO(3)\big)||_{L^\infty(\omega)} \le \frac{1}{2}$$

and then we obtain

$$\text{for a.e.}\quad s \in \Omega_\delta \qquad \det\big(\nabla_x v_\delta(s)\big) > 0. \tag{8.35}$$

Step 2. Strong limit of $\displaystyle\frac{1}{2\delta}\Pi_\delta\Big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\Big)$.

Tanks to the convergences of Lemma 8.1, (8.20) and (8.32) we have

$$||\Pi_\delta\big(\nabla_x v_\delta - \mathbf{R}_\delta\big)||_{(L^2(\Omega))^{3\times 3}} \le C\delta. \tag{8.36}$$

We write the identity $(\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3 = (\nabla_x v_\delta - \mathbf{R}_\delta)^T\mathbf{R}_\delta + \mathbf{R}_\delta^T(\nabla_x v_\delta - \mathbf{R}_\delta) + (\nabla_x v_\delta - \mathbf{R}_\delta)^T(\nabla_x v_\delta - \mathbf{R}_\delta) + (\mathbf{R}_\delta - \mathbf{R})^T\mathbf{R}_\delta + \mathbf{R}^T(\mathbf{R}_\delta - \mathbf{R})$. Due to (8.20), (8.33) and (8.36) we have

$$||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)||_{(L^2(\Omega))^{3\times 3}} \le C\delta. \tag{8.37}$$

In view of (8.18), (8.20), (8.31) we deduce that

$$\begin{cases} \dfrac{1}{\delta}\Pi_\delta\big((\nabla_x v_\delta - \mathbf{R})\mathbf{t}_\alpha\big) \longrightarrow S_3\dfrac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n} \quad \text{strongly in} \quad (L^2(\Omega))^3 \\ \dfrac{1}{\delta}\Pi_\delta\big((\nabla_x v_\delta - \mathbf{R})\mathbf{n}\big) \longrightarrow \dfrac{\partial \overline{w}}{\partial S_3}\mathbf{n} \quad \text{strongly in} \quad (L^2(\Omega))^3 \end{cases} \tag{8.38}$$

Now thanks (8.33) and the strong convergences (8.38) we obtain

$$\frac{1}{\sqrt{\delta}}\Pi_\delta(\nabla_x v_\delta - \mathbf{R}) \longrightarrow 0 \quad \text{strongly in} \quad (L^4(\Omega))^3$$

and then using the above identity, we get

$$\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) \longrightarrow (\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-1} \quad \text{strongly in} \quad (L^2(\Omega))^{3\times 3}, \tag{8.39}$$

where $\widehat{\mathbf{E}}$ is given by (8.14).

Step 3. The $\displaystyle\overline{\lim_{\delta\to 0}}\frac{J(v_\delta)}{\delta^3}$.

Let ε be a fixed positive constant and let θ given by (7.2). We denote χ_δ^θ the characteristic function of the set $A_\delta^\theta = \{s \in \Omega; |||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)(s)||| \ge \theta\}$. Due to (8.37), we have

$$\text{meas}(A_\delta^\theta) \le C\frac{\delta^2}{\theta^2} \tag{8.40}$$

and from (8.35) we have $\det\big(\nabla_x v_\delta(s)\big) > 0$ for a. e. $s \in \Omega_\delta$. Due to (7.2), (7.4) and (8.39) we deduce that

$$\begin{aligned}\overline{\lim_{\delta\to 0}}\int_\Omega \frac{1}{\delta^2}(1-\chi_\delta^\theta)\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\det(\nabla\Phi) \le \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\\ + \varepsilon\int_\Omega |||(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}|||^2\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\end{aligned}$$

where $\widehat{\mathbf{E}}$ is given by (8.14). Thanks to (7.3), (7.4), (8.37), the strong convergence (8.39) and the weak convergence $\dfrac{1}{\delta}\chi_\delta^\theta \rightharpoonup 0$ in $L^2(\Omega)$ we obtain

$$\overline{\lim_{\delta\to 0}}\int_\Omega \frac{1}{\delta^2}\chi_\delta^\theta\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta \det(\nabla\Phi) \le C_1\, \overline{\lim_{\delta\to 0}}\int_\Omega \frac{1}{\delta}\chi_\delta^\theta\,|||\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)|||\,\Pi_\delta \det(\nabla\Phi) = 0$$

Hence for any $\varepsilon > 0$ we get

$$\begin{aligned}\overline{\lim_{\delta\to 0}}\int_\Omega \frac{1}{\delta^2}\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta \det(\nabla\Phi) \le &\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\\ &+\varepsilon\int_\Omega |||(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}|||^2\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\end{aligned}$$

Finally

$$\overline{\lim_{\delta\to 0}}\int_\Omega \frac{1}{\delta^2}\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta \det(\nabla\Phi) \le \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}). \tag{8.41}$$

As far as the contribution of the applied forces is concerned, proceeding as in the proof of the lim-inf condition and using the convergences (8.18) gives

$$\lim_{\delta\to 0}\Big(\int_\Omega f\cdot\Pi_\delta(v_\delta)\Pi_\delta\det(\nabla\Phi) + \int_\Omega g\cdot\frac{\Pi_\delta(v_\delta)}{\delta}\Pi_\delta\det(\nabla\Phi)\Big) = \mathbb{L}(\mathcal{V},\mathbf{R}). \tag{8.42}$$

From (8.41) and (8.42), we conclude that

$$\overline{\lim_{\delta\to 0}}\frac{J(v_\delta)}{\delta^3} \le \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathbb{L}(\mathcal{V},\mathbf{R}) = \mathcal{J}_{NL}\big(\mathcal{V},\mathbf{R},\overline{w}\big).$$

The following theorem summarizes the above results.

Theorem 8.2. *The functional $\mathcal{J}_{NL}$ is the Γ-limit of $\dfrac{J(.)}{\delta^3}$ in the following sense:*

- *consider any sequence of deformations $\big(v_\delta\big)_{0<\delta\le\delta_0}$ belonging to $\mathbf{U}_\delta$ and satisfying*

$$\underline{\lim}_{\delta\to 0}\frac{J(v_\delta)}{\delta^3} < +\infty$$

and let $\big(\mathcal{V}_\delta,\mathbf{R}_{a,\delta},\overline{v}_{a,\delta}\big)$ be the terms of the decomposition of v_δ given by Theorem 3.3. Then there exists $\big(\mathcal{V},\mathbf{R},\overline{w}\big)\in\mathbf{U}_{nlin}$ such that (up to a subsequence)

$$\begin{aligned}&\mathcal{V}_\delta \longrightarrow \mathcal{V} \quad \text{strongly in} \quad \big(H^1(\omega)\big)^3,\\ &\mathbf{R}_{a,\delta} \rightharpoonup \mathbf{R} \quad \text{weakly in} \quad \big(H^1(\omega)\big)^{3\times 3},\\ &\frac{1}{\delta}\Big(\frac{\partial\mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_{a,\delta}\mathbf{t}_\alpha\Big) \rightharpoonup \mathcal{Z}_\alpha \quad \text{weakly in} \quad \big(L^2(\omega)\big)^3,\\ &\frac{1}{\delta^2}\Pi_\delta\overline{v}_{a,\delta} \rightharpoonup \overline{v} \quad \text{weakly in} \quad \big(L^2(\omega;H^1(-1,1))\big)^3\end{aligned}$$

where $\overline{w}$ is defined by (8.13) as a function depending of $\mathbf{R}$, $\mathcal{Z}_\alpha$ and $\overline{v}$. We have

$$\mathcal{J}_{NL}\big(\mathcal{V},\mathbf{R},\overline{w}\big) \le \underline{\lim}_{\delta\to 0}\frac{J(v_\delta)}{\delta^3}$$

• *for any* $(\mathcal{V},\mathbf{R},\overline{w})\in\mathbf{U}_{nlin}$ *there exists a sequence* $(v_\delta)_{0<\delta\le\delta_0}$ *belonging to* $\mathbf{U}_\delta$ *such that*

$$\overline{\lim_{\delta\to 0}}\frac{J(v_\delta)}{\delta^3}\le \mathcal{J}_{NL}(\mathcal{V},\mathbf{R},\overline{w}).$$

Moreover, there exists $(\mathcal{V}_0,\mathbf{R}_0,\overline{w}_0)\in\mathbf{U}_{nlin}$ *such that*

$$m_2^{'}=\lim_{\delta\to 0}\Big(\frac{1}{\delta^3}\inf_{v\in\mathbf{U}_\delta}J(v)\Big)=\mathcal{J}_{NL}(\mathcal{V}_0,\mathbf{R}_0,\overline{w}_0)=\min_{(\mathcal{V},\mathbf{R},\overline{w})\in\mathbf{U}_{nlin}}\mathcal{J}_{NL}(\mathcal{V},\mathbf{R},\overline{w}).\tag{8.43}$$

The next theorem shows that the variable $\overline{w}$ can be eliminated in the minimization problem (8.43).

Theorem 8.3. *Let* $(\mathcal{V}_0,\mathbf{R}_0)$ *be given by Theorem 8.2. The minimum* m_2 *of the functional* $\mathcal{J}_{NL}$ *over* $\mathbf{U}_{nlin}$ *satisfies the following minimization problem:*

$$m_2^{'}=\mathcal{F}_{NL}(\mathcal{V}_0,\mathbf{R}_0)=\min_{(\mathcal{V},\mathbf{R})\in\mathbf{V}_{nlin}}\mathcal{F}_{NL}(\mathcal{V},\mathbf{R}),\tag{8.44}$$

where

$$\begin{aligned}\mathbf{V}_{nlin}=\Big\{(\mathcal{V},\mathbf{R})\in(H^2(\omega))^3\times(H^1(\omega))^{3\times 3}\;\mid\;\mathcal{V}=\phi,\quad \mathbf{R}=\mathbf{I}_3\;\; on\ \gamma_0,\\ \mathbf{R}(s_1,s_2)\in SO(3)\ for\ a.e.\ (s_1,s_2)\in\omega,\quad \frac{\partial\mathcal{V}}{\partial s_\alpha}=\mathbf{R}\mathbf{t}_\alpha\Big\},\end{aligned}$$

and

$$\mathcal{F}_{NL}(\mathcal{V},\mathbf{R})=\int_\omega a_{\alpha\beta\alpha'\beta'}\Big(\frac{\partial\mathbf{R}}{\partial s_\alpha}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\beta\Big)\Big(\frac{\partial\mathbf{R}}{\partial s_\alpha^{'}}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_{\beta'}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})-\mathbb{L}(\mathcal{V},\mathbf{R})$$

where $a_{\alpha\beta\alpha'\beta'}$ *are constants which depend only of the quadratic form* Q *and the vectors* $(\mathbf{t}_1,\mathbf{t}_2,\mathbf{n})$.

Proof of Theorem 8.3. In order to eliminate $\overline{w}$, we fix $(\mathcal{V},\mathbf{R})\in\mathbf{V}_{nlin}$ and we minimize the functional $\mathcal{J}_{NL}(\mathcal{V},\mathbf{R},\cdot)$ over the space

$$\mathbf{W}=\Big\{\overline{w}\in(L^2(\omega;H^1(-1,1)))^3\;\mid\;\int_{-1}^1\overline{w}(s_1,s_2,S_3)dS_3=0\quad\text{for a.e. }(s_1,s_2)\in\omega\Big\}.$$

We first write

$$\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)|\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})|=\int_\Omega\begin{pmatrix}\widehat{Q}_1 & \vdots & \widehat{Q}_2\\ \dots & & \dots\\ \widehat{Q}_2^T & \vdots & \widehat{Q}_3\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{E}}_{11}\\ \widehat{\mathbf{E}}_{12}\\ \widehat{\mathbf{E}}_{22}\\ \widehat{\mathbf{E}}_{13}\\ \widehat{\mathbf{E}}_{23}\\ \widehat{\mathbf{E}}_{33}\end{pmatrix}\cdot\begin{pmatrix}\widehat{\mathbf{E}}_{11}\\ \widehat{\mathbf{E}}_{12}\\ \widehat{\mathbf{E}}_{22}\\ \widehat{\mathbf{E}}_{13}\\ \widehat{\mathbf{E}}_{23}\\ \widehat{\mathbf{E}}_{33}\end{pmatrix}\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})$$

where $\widehat{Q}_1$, $\widehat{Q}_2$ and $\widehat{Q}_3$ are matrices belonging to $(L^\infty(\omega))^{3\times 3}$ moreover $\widehat{Q}_1$ and $\widehat{Q}_3$ are symmetric. They depend of the coefficients of the quadratic form Q and the vectors $(\mathbf{t}_1,\mathbf{t}_2,\mathbf{n})$ and they satisfy

$$\begin{pmatrix}\widehat{Q}_1(s_1,s_2) & \vdots & \widehat{Q}_2(s_1,s_2)\\ \dots & & \dots\\ \widehat{Q}_2^T(s_1,s_2) & \vdots & \widehat{Q}_3(s_1,s_2)\end{pmatrix}\begin{pmatrix}\xi_{11}\\ \xi_{12}\\ \xi_{22}\\ \xi_{13}\\ \xi_{23}\\ \xi_{33}\end{pmatrix}\cdot\begin{pmatrix}\xi_{11}\\ \xi_{12}\\ \xi_{22}\\ \xi_{13}\\ \xi_{23}\\ \xi_{33}\end{pmatrix}\ge c_0|\xi|^2\ \text{ for any }\xi=\begin{pmatrix}\xi_{11}\\ \xi_{12}\\ \xi_{22}\\ \xi_{13}\\ \xi_{23}\\ \xi_{33}\end{pmatrix}\in\mathbb{R}^6\text{ and for a.e. }(s_1,s_2)\in\omega$$

where c_0 is a positive constant.

Through solving simple variational problems (see [18]), we find that the minimum of the functional $\mathcal{J}_{NL}(\mathcal{V}, \mathbf{R}, \cdot)$ over the space $\mathbf{W}$ is obtained with

$$
\begin{pmatrix} \frac{1}{2}\overline{w}(.,.,S_3)\cdot \mathbf{Rt}_1 \\ \frac{1}{2}\overline{w}(.,.,S_3)\cdot \mathbf{Rt}_2 \\ \overline{w}(.,.,S_3)\cdot \mathbf{Rn} \end{pmatrix} = -\Big(S_3^2 - \frac{1}{3}\Big)\widehat{Q}_3^{-1}\widehat{Q}_2^T \begin{pmatrix} \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{Rt}_1 \\ \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{Rt}_2 \\ \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot \mathbf{Rt}_2 \end{pmatrix} \tag{8.45}
$$

Replacing $\begin{pmatrix} \frac{1}{2}\overline{w}(.,.,S_3)\cdot \mathbf{Rt}_1 \\ \frac{1}{2}\overline{w}(.,.,S_3)\cdot \mathbf{Rt}_2 \\ \overline{w}(.,.,S_3)\cdot \mathbf{Rn} \end{pmatrix}$ by its value given above we obtain

$$
\begin{aligned}
&\min_{\overline{w}\in \mathbf{W}} \mathcal{J}_{NL}(\mathcal{V}, \mathbf{R}, \overline{w}) = \mathcal{F}_{NL}(\mathcal{V}, \mathbf{R}) \\
&= \frac{2}{3}\int_\omega \big(\widehat{Q}_1 - \widehat{Q}_2\widehat{Q}_3^{-1}\widehat{Q}_2^T\big) \begin{pmatrix} \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{Rt}_1 \\ \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{Rt}_2 \\ \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot \mathbf{Rt}_2 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{Rt}_1 \\ \dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{Rt}_2 \\ \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot \mathbf{Rt}_2 \end{pmatrix} \det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathbb{L}(\mathcal{V}, \mathbf{R}).
\end{aligned}
$$

The symmetric matrix $\widehat{Q}_1 - \widehat{Q}_2\widehat{Q}_3^{-1}\widehat{Q}_2^T$ belongs to $(L^\infty(\omega))^{3\times 3}$ and moreover it satisfies

$$
\big(\widehat{Q}_1 - \widehat{Q}_2\widehat{Q}_3^{-1}\widehat{Q}_2^T\big)\begin{pmatrix}\xi_{11}\\ \xi_{12}\\ \xi_{22}\end{pmatrix}\cdot\begin{pmatrix}\xi_{11}\\ \xi_{12}\\ \xi_{22}\end{pmatrix} \geq c_0|\xi|^2 \ \text{ for any } \xi = \begin{pmatrix}\xi_{11}\\ \xi_{12}\\ \xi_{22}\end{pmatrix} \in \mathbb{R}^3 \text{ and for a.e. } (s_1, s_2)\in\omega.
$$

□

9. A few remarks.

1. In the case of a St-Venant-Kirchhoff material, the above analysis and a classical energy argument show that if $(v_\delta)_{0<\delta\leq\delta_0}$ is a sequence such that

$$
m_2^{'} = \lim_{\delta\to 0}\frac{J(v_\delta)}{\delta^3},
$$

then there exists a subsequence and $(\mathcal{V}_0, \mathbf{R}_0) \in \mathbf{V}_{nlin}$, which is a solution of Problem (8.44), such that the sequence of the Green-St Venant's deformation tensors satisfies

$$
\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) \longrightarrow (\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{\mathbf{E}}_0(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T} \qquad \text{strongly in} \quad (L^2(\Omega))^{3\times 3},
$$

where $\widehat{\mathbf{E}}_0$ is defined in (8.14) with $\overline{w}_0$ given by (8.45) (replacing $\mathbf{R}$ by $\mathbf{R}_0$).

2. Let us give the explicit expression of the limit energy $\mathcal{F}_{NL}$ in the case where S is a developable surface such that the parametrization ϕ is locally isometric

$$
\forall (s_1, s_2)\in\overline{\omega} \qquad ||\mathbf{t}_\alpha(s_1, s_2)||_2 = 1 \qquad \mathbf{t}_1(s_1, s_2)\cdot\mathbf{t}_2(s_1, s_2) = 0.
$$

We consider a St Venant-Kirchhoff's law for which we have

$$\widehat{W}(F)=\begin{cases}\frac{\lambda}{8}\big(tr(F^TF-\mathbf{I}_3)\big)^2+\frac{\mu}{4}tr\big((F^TF-\mathbf{I}_3)^2\big) & \text{if} \quad \det(F)>0\\ +\infty \qquad \text{if} \qquad \det(F)\le 0,\end{cases}$$

so that $Q=W$. For any $\big(\mathcal{V},\mathbf{R},\overline{w}\big)\in\mathbf{U}_{nlin}$, the expression (8.15) gives

$$\mathcal{J}_{NL}\big(\mathcal{V},\mathbf{R},\overline{w}\big)=\int_\Omega\Big[\frac{\lambda}{2}\big(tr(\widehat{\mathbf{E}})\big)^2+\mu\, tr\big((\widehat{\mathbf{E}})^2\big)\Big]-\mathbb{L}(\mathcal{V},\mathbf{R})$$

where $\widehat{\mathbf{E}}$ is defined by (8.16). It follows that the elimination of $\overline{w}$ in Theorem 8.2 is identical to that of standard linear elasticity (see [19]) hence we have

$$\overline{w}(.,.,S_3)=-\frac{\lambda}{\lambda+2\mu}\Big(S_3^2-\frac{1}{3}\Big)\Big(\frac{\partial\mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1+\frac{\partial\mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2\Big)\mathbf{R}\mathbf{n}$$

and then

$$\mathcal{F}_{NL}\big(\mathcal{V},\mathbf{R}\big)=\frac{2E}{3(1-\nu^2)}\int_\omega\Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\Big(\frac{\partial\mathbf{R}}{\partial s_\alpha}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\beta\Big)^2+\nu\Big(\frac{\partial\mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1+\frac{\partial\mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2\Big)^2\Big]-\mathbb{L}(\mathcal{V},\mathbf{R}).$$

3. It is well known that the constraint $\dfrac{\partial\mathcal{V}}{\partial s_1}=\mathbf{R}\mathbf{t}_1$ and $\dfrac{\partial\mathcal{V}}{\partial s_2}=\mathbf{R}\mathbf{t}_2$ together the boundary conditions are strong limitations on the possible deformation for the limit 2d shell. Actually for a plate or as soon as S is a developable surface, the configuration after deformation must also be a developable surface. In the general case, it is an open problem to know if the set $\mathbf{V}_{nlin}$ contains other deformations than identity mapping or very special isometries (as for example symetries).

Appendix A

In this section the vector space $\mathbb{R}^{n\times n}$ of all matrices with n rows and n is equipped with the Frobenius norm. We set

$$Y=]0,1[^2,\qquad B_3=\Big\{\mathbf{x}\in\mathbb{R}^3\ ;\ ||\mathbf{x}\|_2\le 1\Big\},\qquad S_3=\Big\{\mathbf{x}\in\mathbb{R}^3\ ;\ ||\mathbf{x}\|_2=1\Big\}.$$

We denote $\mathcal{R}_{\mathbf{a},\theta}$ the rotation with axis directed by the vector $\mathbf{a}\in S_3$ and with angle of rotation about this axis $\theta\in\mathbb{R}$,

$$(A.1)\qquad \forall\mathbf{x}\in\mathbb{R}^3,\qquad \mathcal{R}_{\mathbf{a},\theta}(\mathbf{x})=\cos(\theta)\mathbf{x}+(1-\cos(\theta))<\mathbf{x},\mathbf{a}>\mathbf{a}+\sin(\theta)\,\mathbf{a}\wedge\mathbf{x}.$$

Let $\mathbf{R}_0$ and $\mathbf{R}_1$ be two matrices in $SO(3)$. Matrix $\mathbf{R}_0$ represent the rotation $\mathcal{R}_{\mathbf{a}_0,\theta_0}$ and matrix $\mathbf{R}_1$ represent the rotation $\mathcal{R}_{\mathbf{a}_1,\theta_1}$. The linear transformation in $\mathbb{R}^3$

$$x\longmapsto 2\big(\sin(\theta_1)\mathbf{a}_1-\sin(\theta_0)\mathbf{a}_0\big)\wedge x$$

has for matrix $\mathbf{R}_1-\mathbf{R}_0-(\mathbf{R}_1-\mathbf{R}_0)^T$ and we have

$$\big\|\sin(\theta_1)\mathbf{a}_1-\sin(\theta_0)\mathbf{a}_0\big\|_2=\frac{1}{2\sqrt{2}}|||\mathbf{R}_1-\mathbf{R}_0-(\mathbf{R}_1-\mathbf{R}_0)^T|||\le\frac{1}{\sqrt{2}}|||\mathbf{R}_1-\mathbf{R}_0|||.$$

To any matrix $\mathbf{R}$ in $SO(3)$ we associate the vector $\mathbf{b} = \sin(\theta)\mathbf{a}$ where $\mathbf{R}$ is the matrix of the rotation $\mathcal{R}_{\mathbf{a},\theta}$. This map is continuous from $SO(3)$ into B_3 and from the above inequality, we obtain

$$\|\mathbf{b}\|_2 \le \frac{1}{\sqrt{2}}|||\mathbf{R} - \mathbf{I}_3|||.$$

If $\cos(\theta) \neq -1$, using the vector $\mathbf{b}$ we can write the rotation $\mathcal{R}_{\mathbf{a},\theta}$ as

$$(A.2) \qquad \forall \mathbf{x} \in \mathbb{R}^3, \qquad \mathcal{R}_{\mathbf{a},\theta}(\mathbf{x}) = \cos(\theta)\mathbf{x} + \frac{1}{1+\cos(\theta)} < \mathbf{x}, \mathbf{b} > \mathbf{b} + \mathbf{b} \wedge \mathbf{x}.$$

Let $\mathbf{R}_0$ and $\mathbf{R}_1$ be two matrices in $SO(3)$ such that

$$|||\mathbf{R}_0 - \mathbf{R}_1||| < 2\sqrt{2}.$$

Now we define a path $\mathbf{f}$ from $\mathbf{R}_0$ to $\mathbf{R}_1$:

• if $\mathbf{R}_1 = \mathbf{R}_0$ we choose the constant function $\mathbf{f}(t) = \mathbf{R}_0$, $t \in [0, 1]$,

• if $\mathbf{R}_1 \neq \mathbf{R}_0$, we set $\mathbf{R}_2 = \mathbf{R}_0^{-1}\mathbf{R}_1$, there exists a unique pair $(\mathbf{a}_2, \theta_2) \in S_3 \times]0, \pi[$ such that the matrix $\mathbf{R}_2$ represent the rotation with axis directed by the vector $\mathbf{a}_2$ and with the angle θ_2. We consider the rotations field $\mathcal{R}_{\mathbf{a}(t),\theta(t)}$ given by formula (A.1) where

$$\mathbf{a}(t) = \mathbf{a}_2, \qquad \theta(t) = t\theta_2, \qquad t \in [0, 1]$$

and we define $\mathbf{f}(t)$ as the matrix of the rotation $\mathcal{R}_0 \circ \mathcal{R}_{\mathbf{a}(t),\theta(t)}$ where $\mathcal{R}_0$ is the rotation with matrix $\mathbf{R}_0$.

Lemma A.1. *The path $\mathbf{f}$ belongs to $W^{1,\infty}(0, 1; SO(3))$ and satisfies*

$$(A.3) \qquad \begin{cases} \mathbf{f}(0) = \mathbf{R}_0, \qquad \mathbf{f}(1) = \mathbf{R}_1, \qquad \Big\|\dfrac{d\mathbf{f}}{dt}\Big\|_{(L^\infty(0,1))^9} \le \dfrac{\pi}{2}|||\mathbf{R}_1 - \mathbf{R}_0||||, \\ |||\mathbf{R}_0 - \mathbf{f}(t)||| \le |||\mathbf{R}_0 - \mathbf{R}_1|||. \end{cases}$$

Proof One has

$$\Big\|\frac{d\mathbf{f}}{dt}\Big\|_{(L^\infty(0,1))^9} = \sqrt{2}\theta_2 \le \frac{\pi}{2}|||\mathbf{R}_2 - \mathbf{I}_3||| = \frac{\pi}{2}|||\mathbf{R}_1 - \mathbf{R}_0|||.$$

Moreover

$$|||\mathbf{R}_0 - \mathbf{f}(t)||| = |||I_3 - \mathbf{R}_0^{-1}\mathbf{f}(t)||| = 2\sqrt{2}\sin\Big(\frac{\theta_2 t}{2}\Big) \le 2\sqrt{2}\sin\Big(\frac{\theta_2}{2}\Big) = |||I_3 - \mathbf{R}_2||| = |||\mathbf{R}_0 - \mathbf{R}_1||||.$$

□

Lemma A.2. *Let $\mathbf{R}_{00}$, $\mathbf{R}_{01}$, $\mathbf{R}_{10}$ and $\mathbf{R}_{11}$ be four matrices belonging to $SO(3)$ and satisfying*

$$(A.4) \qquad |||\mathbf{R}_{10} - \mathbf{R}_{00}||| \le \frac{1}{2}, \qquad |||\mathbf{R}_{01} - \mathbf{R}_{00}||| \le \frac{1}{2}, \qquad |||\mathbf{R}_{11} - \mathbf{R}_{01}||| \le \frac{1}{2}, \qquad |||\mathbf{R}_{11} - \mathbf{R}_{10}||| \le \frac{1}{2}.$$

There exists a function $\mathbf{R} \in W^{1,\infty}(Y; SO(3))$ such that

$$(A.5) \qquad \begin{cases} \mathbf{R}(0,0) = \mathbf{R}_{00}, \qquad \mathbf{R}(0,1) = \mathbf{R}_{01}, \qquad \mathbf{R}(1,0) = \mathbf{R}_{10}, \qquad \mathbf{R}(1,1) = \mathbf{R}_{11}, \\ ||\nabla\mathbf{R}||_{(L^\infty(Y))^9} \le C\big\{|||\mathbf{R}_{10} - \mathbf{R}_{00}|||| + |||\mathbf{R}_{01} - \mathbf{R}_{00}|||| + |||\mathbf{R}_{11} - \mathbf{R}_{01}|||| + |||\mathbf{R}_{11} - \mathbf{R}_{10}||||\big\}. \end{cases}$$

and where the functions $x_1 \longrightarrow \mathbf{R}(x_1,0)$, $x_1 \longrightarrow \mathbf{R}(x_1,1)$, $x_2 \longrightarrow \mathbf{R}(0,x_2)$ *and* $x_2 \longrightarrow \mathbf{R}(1,x_2)$ *are paths given by Lemma A.1.*

Proof. We denote

$\mathbf{f}_{00,01}$ the path from $\mathbf{R}_{00}$ to $\mathbf{R}_{01}$,

$\mathbf{f}_{01,11}$ the path from $\mathbf{R}_{01}$ to $\mathbf{R}_{11}$,

$\mathbf{f}_{00,10}$ the path from $\mathbf{R}_{00}$ to $\mathbf{R}_{10}$ and

$\mathbf{f}_{01,11}$ the path from $\mathbf{R}_{01}$ to $\mathbf{R}_{11}$ given by Lemma A.

From Lemma A.1, we have

$$\forall t \in [0,1], \qquad \begin{cases} |||\mathbf{f}_{00,01}(t) - \mathbf{R}_{00}||| \leq 1, & |||\mathbf{f}_{01,11}(t) - \mathbf{R}_{00}||| \leq 1, \\ |||\mathbf{f}_{00,10}(t) - \mathbf{R}_{00}||| \leq 1, & |||\mathbf{f}_{01,11}(t) - \mathbf{R}_{00}||| \leq 1. \end{cases}$$

For any $t \in [0,1]$,

to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{00,01}(t)$ we associate the vector $\mathbf{b}_{00,01}(t)$,

to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{01,11}(t)$ we associate the vector $\mathbf{b}_{01,11}(t)$,

to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{00,10}(t)$ we associate the vector $\mathbf{b}_{00,10}(t)$ and

to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{01,11}(t)$ we associate the vector $\mathbf{b}_{01,11}(t)$.

Let $\mathbf{b}$ be the vectors field defined by

$$\mathbf{b}(x_1,x_2) = \begin{cases} \mathbf{b}_{00,10}(0)\big(= \mathbf{b}_{00,01}(0)\big) \qquad \text{if} \qquad (x_1,x_2) = (0,0), \\ \dfrac{x_1}{x_1+x_2}\mathbf{b}_{00,10}(x_2) + \dfrac{x_2}{x_1+x_2}\mathbf{b}_{00,01}(x_1) \quad \text{if } 0 < x_1+x_2 \leq 1 \\ \dfrac{1-x_1}{2-x_1-x_2}\mathbf{b}_{10,11}(x_2) + \dfrac{1-x_2}{2-x_1-x_2}\mathbf{b}_{01,11}(x_1) \quad \text{if } 1 \leq x_1+x_2 < 2 \\ \mathbf{b}_{01,11}(1)\big(= \mathbf{b}_{10,11}(1)\big) \qquad \text{if} \qquad (x_1,x_2) = (1,1). \end{cases}$$

This function belongs to $\big(W^{1,\infty}(Y)\big)^3$ and satisfies

$$\forall (x_1,x_2) \in \overline{Y}, \qquad \|\mathbf{b}(x_1,x_2)\|_2 \leq \frac{1}{\sqrt{2}}.$$

Now we introduce the rotations field $\mathcal{R}(x_1,x_2)$ given by formula (A.2) where $\mathbf{b}(x_1,x_2)$ is defined above and where

$$\theta(x_1,x_2) = \arccos\sqrt{1- < \mathbf{b}(x_1,x_2), \mathbf{b}(x_1,x_2) >}, \qquad (x_1,x_2) \in \overline{Y}.$$

Let $\mathbf{R}(x_1,x_2)$ be the matrix of the rotation $\mathcal{R}_{00} \circ \mathcal{R}(x_1,x_2)$ where $\mathcal{R}_{00}$ is the rotation with matrix $\mathbf{R}_{00}$. It is easy to check that $\mathbf{R}$ satisfies the conditions (A.5). □

Corollary of Lemma A.2. Let $\mathbf{R}_a$ be the Q_1 interpolate of the matrices $\mathbf{R}_{00}$, $\mathbf{R}_{01}$, $\mathbf{R}_{10}$ and $\mathbf{R}_{11}$. There exists a strictly positive constant C such that

$$||\mathbf{R} - \mathbf{R}_a||_{(L^2(Y)^9} \leq C\big\{|||\mathbf{R}_{10} - \mathbf{R}_{00}|||| + |||\mathbf{R}_{11} - \mathbf{R}_{01}|||| + |||\mathbf{R}_{01} - \mathbf{R}_{00}|||| + |||\mathbf{R}_{11} - \mathbf{R}_{10}||||\big\}.$$

□

Appendix B

Lemma B.1. *Let $\mathcal{Q}_m$ be the positive definite quadratic form defined on the space $\mathbb{R}^3 \times \mathbb{R}^3 \times \big(L^2(-1,1)\big)^3$ by*

$$\forall(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \big(L^2(-1,1)\big)^3, \qquad \mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}) = \int_{-1}^{1} \mathbf{A} \begin{pmatrix} S_3\mathbf{a}_1 + \mathbf{b}_1 \\ S_3\mathbf{a}_2 + \mathbf{b}_2 \\ S_3\mathbf{a}_3 + \mathbf{b}_3 \\ \mathbf{c}_1(S_3) \\ \mathbf{c}_2(S_3) \\ \mathbf{c}_3(S_3) \end{pmatrix} \cdot \begin{pmatrix} S_3\mathbf{a}_1 + \mathbf{b}_1 \\ S_3\mathbf{a}_2 + \mathbf{b}_2 \\ S_3\mathbf{a}_3 + \mathbf{b}_3 \\ \mathbf{c}_1(S_3) \\ \mathbf{c}_2(S_3) \\ \mathbf{c}_3(S_3) \end{pmatrix} dS_3$$

where $\mathbf{A}$ is a symmetric positive definite constant 6×6 matrix. For any $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \big(L^2(-1,1)\big)^3$, there exists $\mathbf{d}$ in $\mathbf{R}^3$ which depends only on $\mathbf{b}$ and $\mathbf{A}$ such that

$$\mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}) \geq \mathcal{Q}_m(\mathbf{a},0,\mathbf{c}-\mathbf{d}).$$

Proof. We write

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \vdots & \mathbf{A}_2 \\ \cdots & & \cdots \\ \mathbf{A}_2^T & \vdots & \mathbf{A}_3 \end{pmatrix}$$

where $\mathbf{A}_1$ and $\mathbf{A}_3$ are symmetric positive definite 3×3 matrices. We set

$$\mathbf{d} = -\mathbf{A}_3^{-1}\mathbf{A}_2^T\mathbf{b}.$$

By a straightforward calculation and using the fact that $\displaystyle\int_{-1}^{1} S_3 dS_3 = 0$, we obtain

$$\mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathcal{Q}_m(\mathbf{a},0,\mathbf{c}-\mathbf{d}) + \mathcal{Q}_m(0,\mathbf{b},\mathbf{d}).$$

The positivity of $\mathcal{Q}$ then gives the result. $\square$

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